Taylor expansions in powers of time of Lagrangian and Eulerian two-point two-time velocity correlations in turbulence

Yukio Kaneda, Takashi Ishihara, and Koji Gotoh
Department of Computational Science and Engineering, Graduate School of Engineering, Nagoya University, Chikusa-ku, Nagoya 464-8603, Japan

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A method is developed for generating the Taylor expansions in powers of the time difference of the Lagrangian and Eulerian two-point two-time velocity correlations in turbulence. The expansions are based on the Taylor series of the Eulerian and Lagrangian velocity fields subject to given dynamics along with initial and boundary conditions. The lowest few coefficients in the expansions enable us to construct approximations to the correlations. An application of the method to turbulence obeying the Navier-Stokes dynamics yields approximations, particularly Padé approximations that agree well with direct numerical simulations of homogeneous isotropic turbulence at moderate Reynolds numbers. The ratios of the second-order to the zeroth-order coefficients of the Taylor series of the Lagrangian and Eulerian correlations give, respectively, the estimates for the Lagrangian and Eulerian micro time scales \( \tau_L \) and \( \tau_E \). An analysis of a high resolution \((512^3\) grid points) direct numerical simulation database at large Reynolds number suggests the scalings \( \tau_L \propto k^{-2/3} \) and \( \tau_E \propto k^{-1} \) for wave numbers \( k \) in the inertial subrange. The role of flow structures in turbulence in determining the time scales is also discussed. © 1999 American Institute of Physics.

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I. INTRODUCTION

Turbulence is a nonlinear dissipative phenomenon involving interactions that cover a wide range of spatial and temporal scales. Two-point two-time correlations or their spectra are some of the simplest measures characterizing the statistical state of such turbulence, and may play important roles in the study of turbulence. In fact, the Eulerian two-point two-time velocity correlation, together with the function representing the response of the turbulence to an infinitesimal disturbance, play the key roles in the direct interaction approximation (DIA), as well as in various Eulerian statistical closure theories of turbulence, whereas the Lagrangian two-point two-time velocity correlations are regarded as fundamental quantities in Lagrangian closure approximations such as the abridged Lagrangian-history DIA (ALH/DIA) and the Lagrangian renormalized approximation (LRA).

The study of two-point two-time correlations is interesting not only from closure theoretical, but also from practical points of view. For example, as is well known, the turbulent single-particle, as well as pair-particle diffusivities, are determined by the Lagrangian two-time single- and two-point correlations. It is therefore expected that the study of these correlations, and developing reasonable approximations to them, may contribute to the understanding of transport phenomena such as heat and mass transfer in turbulence.

In this paper we use the method of Taylor expansions of two-time correlations in powers of the time difference between the two times. The method of Taylor expansion in powers of time in Eulerian coordinates has been used in the studies of the Taylor-Green flow (cf. Refs. 4–6 and the references cited therein). Kraichnan studied a method for constructing approximations to statistical functions which uses the Taylor series in time and expansions of the Fourier transform of the unknown function by a set of orthonormal functions, and showed that the method yields results that agree well with kinematical simulations for the Lagrangian single-point velocity correlation and the eddy diffusivity for marked particles in given artificial random velocity fields. Dannevik used the Taylor expansions in time of Eulerian statistical functions in the DIA equations, from the viewpoint of economizing the computations needed to solve the DIA equations, and showed that Padé approximations constructed from the Taylor series may yield reasonable approximate solutions of the DIA.

The coefficients in the Taylor series in real turbulence are determined by the dynamics governing the fluid motion, as well as by the boundary and initial conditions. The expansion in powers of time is conceptually straightforward, particularly in Eulerian coordinates. However, it is practically difficult to calculate analytically the coefficients satisfying the given equations of the dynamics, especially for general boundary and initial conditions. In this paper, we use a computational method to generate the Taylor series of the Eulerian and Lagrangian velocity fields. The method is applicable to a wide class of dynamics, namely those which may be written in the form

\[
\frac{\partial}{\partial t} \mathbf{u} = \mathcal{M}: \mathbf{uu} + \mathcal{L} \mathbf{u} + \mathbf{f}, \tag{1}
\]

where \( \mathcal{M} \), \( \mathcal{L} \) and \( \mathbf{f} \) stand, respectively, for bilinear and linear operators and an external force field. Here \( \mathbf{u} \) need not be restricted to a velocity field. It may also stand for fields in-
cluding temperature, density, magnetic field, etc. If one has any code to compute the right-hand side (r.h.s.) of (1), then one can apply the method to generate the Taylor expansions of the Eulerian and Lagrangian velocity fields.

In a recent letter,\(^7\) the method was applied to construct the Taylor series of the Eulerian and Lagrangian single-point correlations in turbulence obeying the Navier-Stokes dynamics, and it was shown that the Padé approximations constructed from the Taylor coefficients are in good agreement with direct numerical simulation (DNS) at moderate Reynolds numbers. In Secs. II and III, we extend the study to the two-point correlations and test the performance of approximations constructed from the Taylor series. Although the notations and the basic idea to compute the Taylor series of the velocity field are the same as in Ref. 9, some more details of the idea and method will be presented for the reader’s convenience. It is shown in Sec. III that approximations using the lowest few coefficients in the Taylor series, including those proposed by Kraichnan and Padé approximations, may yield results in reasonable agreement with DNS of homogeneous isotropic turbulence at moderate Reynolds numbers. Low-order terms in the Taylor series, e.g., up to second-order ones, are much easier to compute than higher-order ones. It is shown that even these lowest-order terms may be sufficient for obtaining a rough estimate for the characteristic time scales in the cases studied here. In Sec. IV, we consider the Eulerian time microscale \(\tau_E\) and the Lagrangian one \(\tau_L\), derived from the zeroth- and second-order coefficients of the Taylor series of the Eulerian and Lagrangian two-time spectra. An analysis of a database of DNS with 512\(^3\) grid points suggests that the time microscales scale with the wave number \(k\) as \(\tau_E \propto k^{-1}\) and \(\tau_L \propto k^{-2/3}\) in the inertial subrange, and that \(\tau_L(k)\) increases rapidly with \(k\) and much larger than \(\tau_E(k)\) at large \(k\). The role of flow structures, as represented by the non-Gaussianity of the probability distribution function (PDF) of the velocity field, in determining the two-time correlations is discussed in Sec. V.

II. TAYLOR EXPANSIONS IN POWERS OF TIME

Let us consider the Taylor expansions in powers of time \(t\) of the Eulerian and Lagrangian velocity fields

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u},
\]

where \(\nu\) is the kinematic viscosity. Since the pressure \(p\) may be eliminated from (4) by the use of the incompressibility condition \(\nabla \cdot \mathbf{u} = 0\) under appropriate boundary conditions, we may write (4) symbolically as (1) with \(\mathbf{F} = 0\) after the elimination of the pressure term.

Regarding the Eulerian time derivatives in (2), it is readily shown that

\[
\mathbf{u}^{(n+1)} = \sum_{m=0}^{n} n C_m M^n \mathbf{u}^{(m)} + \mathcal{L} \mathbf{u}^{(n)},
\]

by using (1) recursively, where \(n C_m = n!/[m!(n-m)!]\) and \(\mathbf{u}^{(0)} = \mathbf{u}(x,t_0)\).

Regarding the Lagrangian time derivatives in (3), it is shown that

\[
\mathbf{v}^{(n)}(x) = \left[ \frac{\partial}{\partial t} + (\mathbf{u}(x,t+t_0) \cdot \nabla) \right]^n \mathbf{u}(x,t+t_0) \bigg|_{t=0}
\]

\[
= (\mathcal{T} + \mathcal{U})^n \mathbf{u},
\]

by considering the meaning of the Lagrangian time derivative, where the operators \(\mathcal{T}\) and \(\mathcal{U}\) are defined by

\[
\mathcal{T} \mathbf{a} = \frac{\partial}{\partial t} \mathbf{a}, \quad \mathcal{U} \mathbf{a} = (\mathbf{u} \cdot \nabla) \mathbf{a}.
\]

Equation (6) may be also justified by noting that

\[
\mathbf{v}(x,t',t) = \int \mathbf{u}(y,t') \psi(y,t;x,t') d^3y,
\]

and

\[
\frac{\partial}{\partial t} \int g(y,t) \psi(y,t;x,s) d^3y
\]

\[
= \int \left[ \frac{\partial g(y,t)}{\partial t} \psi(y,t;x,s) + g(y,t) \frac{\partial}{\partial t} \psi(y,t;x,s) \right] d^3y
\]

\[
= \int \frac{\partial g(y,t)}{\partial t} + (\mathbf{u}(y,t) \cdot \nabla) g(y,t) \psi(y,t;x,s) d^3y,
\]

for any \(g(y,t)\), where \(\psi\) is the Lagrangian position function defined by

\[
\psi(y,t;x,s) = \delta^3(y - \mathbf{r}(x,s;t)),
\]

and obeys

\[
\frac{\partial}{\partial t} + (\mathbf{u}(x,t) \cdot \nabla) \psi(x,t;x',t') = 0,
\]

\[
\psi(x,t;x',t) = \delta^3(x - x'),
\]

in which \(\mathbf{r}(x,t;s)\) is the position vector at time \(s\) of the particle that was at position \(x\) at time \(t\). Equation (8) is derived by (9) and partial integrations with respect to \(y\).

As an illustration, the first few equations in the recursion are given below,

\[
\mathbf{v}^{(0)} = \mathbf{u}^{(0)},
\]

\[
\mathbf{v}^{(1)} = (7\mathbf{u} + \mathcal{U} \mathbf{u})|_{t=0} = \mathbf{u}^{(1)} + (\mathbf{u}^{(0)} \cdot \nabla) \mathbf{u}^{(0)},
\]

\[
\mathbf{v}^{(2)} = \sum_{m=0}^{1} \frac{n}{n!} \mathbf{v}^{(m)}. \]
\[ \mathbf{v}^{(2)} = (T^2 \mathbf{u} + \mathcal{U} \mathbf{u} + T \mathcal{L} \mathbf{u} + \mathcal{L} \mathbf{u}) |_{r=0} \]
\[ = \mathbf{u}^{(2)} + (\mathbf{u}^{(0)} \cdot \nabla) \mathbf{u}^{(1)} + [(\mathbf{u}^{(1)} \cdot \nabla) \mathbf{u}^{(0)} + (\mathbf{u}^{(0)} \cdot \nabla) \mathbf{u}^{(1)}] + (\mathbf{u}^{(0)} \cdot \nabla) (\mathbf{u}^{(0)} \cdot \nabla) \mathbf{u}^{(0)} \ldots \ldots \]

In order to see more clearly the structure of the terms involved, it is convenient to use an abbreviated notation. Let us write
\[ (u^{(i-1)} \cdot \nabla)(u^{(i-1)} \cdot \nabla)(u^{(k-1)} \cdot \nabla)(u^{(m-1)}) |_{r=0} = (i,j,k,...,m). \] (11)

Then, the above expressions may be rewritten as
\[ v^{(0)} = (1), \]
\[ v^{(1)} = \mathcal{T} (1) + \mathcal{U} (1), \]
\[ v^{(2)} = \mathcal{T} \mathcal{T} (1) + \mathcal{T} \mathcal{L} (1) + \mathcal{U} \mathcal{T} (1) + \mathcal{U} \mathcal{L} (1). \]

It is observed that
\[ \mathcal{T} (1) = (2), \quad \mathcal{U} (1) = (1,1), \]
\[ \mathcal{T} \mathcal{T} (1) = (2) = (3), \quad \mathcal{T} \mathcal{L} (1) = (1,1) = (1,2) + (2,1), \]
\[ \mathcal{U} \mathcal{T} (1) = \mathcal{U} (2) = (1,2), \quad \mathcal{U} \mathcal{L} (1) = \mathcal{L} (1,1) = (1,1,1). \]

It can be shown that the operators \( \mathcal{T} \) and \( \mathcal{U} \) have the following properties:
\[ \mathcal{T} (i,j,...,k) = (i+1,j,...,k) + (i,j+1,...,k) + \cdots + (i,j,...,k+1), \] (12a)
\[ \mathcal{U} (i,j,...,k) = (i,j,...,k), \] (12b)
and \( v^{(n)} \) is a linear combination of terms such as \( (i,j,...,k) \) satisfying
\[ i + j + \cdots + k = n + 1. \] (13)

There are \( 2^n \) kinds of \( (i,j,...,k) \) terms satisfying (13), such as
\[ (n+1), (n,1), (n-1,2), (n-1,1,1), \ldots, \]

because the problem of finding the sets of positive integers \( i,j,...,k \) satisfying (13) is equivalent to finding the ways of partitioning, along one spatial dimension, a sequence of \( n+1 \) “balls” by “walls.” For example, for \( n = 3 \) we may use the following correspondence rule, where the symbols “-” and “|” stand for one “ball” and one “wall,” respectively,
\[ (4) \rightarrow (\cdot, \cdot, \cdot, \cdot) \rightarrow (000), \]
\[ (3,1) \rightarrow (\cdot, \cdot, \cdot | \cdot) \rightarrow (001), \]
\[ (2,1,1) \rightarrow (\cdot, \cdot, | \cdot, \cdot) \rightarrow (011), \]
\[ \ldots \ldots \]

Thus there is one-to-one correspondence between the above sets of integers and the binary numbers for \( n \geq 1 \). Using such binary numbers, we may write (12) as
\[ \mathcal{T} [s_1 s_2 s_3 \cdots s_n] = [0 s_1 s_2 s_3 \cdots s_n] \]
\[ + \sum_{i=1}^{n} [ \text{replace } s_i \text{ to } 10 \text{ if } s_i = 1]. \] (14a)
\[ \mathcal{U} [s_1 s_2 s_3 \cdots s_n] = [1 s_1 s_2 s_3 \cdots s_n], \] (14b)

where \([s_1 s_2 s_3 \cdots s_n]\) denotes a binary number with \( s_i = 0 \) or \( 1 \) \( (i=1,2,\ldots,n) \). One may express \( (T+\mathcal{U})^n (1) = (T+\mathcal{U})^{\cdot n} \) as a linear combination of the binary numbers. The coefficients in the combination can be generated by using the rule (14) iteratively.

From (5) and (6), we can compute \( (\mathbf{u}^{(n)},\mathbf{v}^{(n)}) \), for \( n = 1,2,3, \ldots \), successively from any given \( \mathbf{u}^{(0)} \), provided that \( \mathcal{M}: a \mathbf{b} = (\cdot \cdot \cdot \cdot \cdot \cdot \cdot) \) and \( \mathcal{C} \mathbf{a} = \mathbf{c} \mathbf{a} \) can be computed for any \( \mathbf{a} \) and \( \mathbf{b} \). The number of nonlinear operations \( \mathcal{M}: a \mathbf{b} \) and \( \mathcal{C} \mathbf{a} \) required to compute \( \mathbf{u}^{(n)} \) and \( \mathbf{v}^{(n)} \) can be shown to grow as \( O(n) \) and \( O(2^n) \) for large \( n \), respectively.

In practice, it is difficult to perform the operations \( \mathcal{M} \) and \( \mathcal{C} \) exactly, and we use below the Fourier spectral method in which the vector field \( \mathbf{a} \) and the operators \( \mathcal{M}, \mathcal{L}, \) and \( \mathcal{C} \) are approximated by \( \mathbf{a} = \mathcal{F} \mathbf{a} \) and \( \mathcal{M} = \mathcal{F} \mathcal{M}, \mathcal{L} = \mathcal{F} \mathcal{L}, \) and \( \mathcal{C} = \mathcal{F} \mathcal{C} \), respectively, where \( \mathcal{F} \) denotes the filtering operation of discarding Fourier modes of wave number higher than some \( K_{max} \), and \( \mathcal{L} = \mathcal{F} \mathcal{L} = \mathcal{L} \) in the spectral method, under periodic boundary conditions. This implies that the original dynamics (1) with \( \mathbf{f} = 0 \) is approximated by
\[ \frac{\partial}{\partial t} \mathbf{u} = \mathcal{M} : \mathbf{u} \mathbf{u} + \mathbf{u}. \] (15)

The Taylor expansion series (2) obtained by (5) with \( \mathcal{M} \) replaced by \( \mathcal{M} \) exact for the field obeying (15) instead of (1). The series (2) and (3) are error-free in \( t \), since there is no time discretization. As discussed in Ref. 9, the individual trajectory in phase space of a solution of (1) may in general be different in a strict sense from that of (15), and our use of the filtered field \( \mathbf{u} \) and operators \( \mathcal{M}, \mathcal{C} \) is based on the assumption that solutions of (15) can still represent some features, particularly certain statistical properties, of the exact dynamics (1). A similar assumption also underlies any DNS of turbulence: one can treat only a finite number of modes and solves (15) rather than (1) (under appropriate definitions of the operators \( \mathcal{F}, \mathcal{M} \) and \( \mathcal{L} \)). From a pragmatic point of view, the possible effects of the filtering or the resolution may be estimated, e.g., by checking possible dependences on the truncation wave number \( K_{max} \).

From the expansions (2) and (3), we can generate the Taylor series of the Eulerian and Lagrangian two-point two-time correlations \( R_E \) and \( R_L \) as
\[ R_E(t+t_0, \mathbf{r}) = (\mathbf{u}(\mathbf{x}+\mathbf{r}, t+t_0) \cdot \mathbf{u}(\mathbf{x}, t_0)) = \sum_n \frac{r^n}{n!} c_E^{(n)}(\mathbf{r}), \] (16)
\[ R_L(t+t_0, \mathbf{r}) = (\mathbf{v}(\mathbf{x}+\mathbf{r}, t+t_0) \cdot \mathbf{v}(\mathbf{x}, t_0)) \]
\[ = \sum_n \frac{r^n}{n!} c_L^{(n)}(\mathbf{r}), \] (17)
where
\[ c_E^{(n)}(\mathbf{r}) = \langle (\mathbf{u}^n)(\mathbf{x} + \mathbf{r}) \cdot \mathbf{u}(\mathbf{x}) \rangle, \]
\[ c_L^{(n)}(\mathbf{r}) = \langle (\mathbf{v}^n)(\mathbf{x} + \mathbf{r}) \cdot \mathbf{u}(\mathbf{x}) \rangle, \]
(18)
the brackets denote the ensemble average, and we omit writing
the argument \( \mathbf{x} \) at will. In (18), we have used \( \mathbf{v}(\mathbf{x},t_0; t) = \mathbf{u}(\mathbf{x}, t_0) = \mathbf{u}(\mathbf{x}) \).

In the Fourier space defined by
\[ f(\mathbf{x}) = \sum_k \hat{f}(k) \text{exp}(i\mathbf{k} \cdot \mathbf{x}), \]
under periodic boundary conditions, (16) and (17) yield
\[ \hat{R}_E(t, t_0, \mathbf{k}) = \langle \hat{\mathbf{u}}(\mathbf{k}, t + t_0) \cdot \hat{\mathbf{u}}(-\mathbf{k}, t_0) \rangle = \sum_n \frac{t^n}{n!} \hat{c}_E^{(n)}(\mathbf{k}), \]
\[ \hat{R}_L(t, t_0, \mathbf{k}) = \langle \hat{\mathbf{v}}(\mathbf{k}, t + t_0) \cdot \hat{\mathbf{v}}(-\mathbf{k}, t_0; t) \rangle = \sum_n \frac{t^n}{n!} \hat{c}_L^{(n)}(\mathbf{k}), \]
(19)
for homogeneous turbulence, where
\[ \hat{c}_E^{(n)}(\mathbf{k}) = \langle \hat{\mathbf{u}}^n(\mathbf{k}) \cdot \hat{\mathbf{u}}(-\mathbf{k}) \rangle, \]
\[ \hat{c}_L^{(n)}(\mathbf{k}) = \langle \hat{\mathbf{v}}^n(\mathbf{k}) \cdot \hat{\mathbf{u}}(-\mathbf{k}) \rangle. \]
(20)

III. APPROXIMATIONS BASED ON THE TAYLOR EXPANSIONS

Since the number of nonlinear operations and machine memory
necessary for the computation of the coefficients \( c_E^{(n)} \) and \( c_L^{(n)} \) increases rapidly with \( n \), the number of coefficients
that can be calculated in practice is limited. One must truncate the Taylor expansions (2) and (3) at some finite order. Suppose one wishes to construct an approximation, or
to get an idea of the correlations from such a truncated Taylor series: one may then expect that the truncated Taylor series may give a good approximation for small \( t \). One may also take into account that the correlations must tend to 0 as \( t \to \infty \) under fairly weak assumption(s) such as the ergodicity, and that the correlations are in general so small, for sufficiently large \( t \), that quantities of practical interest, such as the Lagrangian integral time scale that determines the turbulent diffusivity, may be insensitive to the exact behavior of the correlations at large \( t \). One is then led to try an approximation satisfying the following two conditions:

(a) it is compatible with the Taylor series for small \( t \), and
(b) it tends to 0 as \( t \to \infty \).

The truncated Taylor series itself clearly satisfies (a) but not
(b), because the absolute value of any polynomial of finite order, however high the order may be, tends to infinity as \( t \to \infty \). A simple approximation satisfying (a) and (b) may be obtained by a Padé approximation. The evaluations of the coefficients \( c_E^{(n)} \) and \( c_L^{(n)} \) from \( n = 0 \) to \( n = N \) enable one to construct Padé approximations for the correlations,
\[ R^S(t | K,M)(t) = \frac{P_K(t)}{P_M(t)}, \]
(22)
The values of the Pade’ approximations are obtained according to the following algorithm.

(i) Compute $u^{(n)}$ and $v^{(n)}$, $n = 1, 2, 3, \ldots$, according to (5) and (6), and by substituting the Eulerian velocity field at time $t_0$ for $u_0$.

(ii) Take the averages in (18) for the computation of the coefficients $c_S^{(n)}$.

(iii) Compute $c_S^{(n)}$, $n = 1, 2, 3, \ldots$, by (24).

(iv) Construct the Padé approximations (22) compatible with the Taylor series (23).

We have computed $u^{(n)}$ and $v^{(n)}$ up to $n = 6$ as in Ref. 9. As noted there, the polynomials in the denominators of the Padé approximations can have real zeros, and the approximations would then blow up near such points. We omit plotting Padé approximations exhibiting such blow up within the observed range of $t$, and show values of only some typical Padé approximations. Figures 1 and 2 show that the agreement between the DNS and the Padé approximations is satisfactory.

In order to see the possible effect of the truncation in the wave-vector space in computing $c_S^{(n)}$ and $c_L^{(n)}$ by (5) and (6), we tried the computation with doubling the cut-off wave number $K_{\text{max}}$ (i.e., changing $K_{\text{max}}$ from 21 to 42). The computation suggests that the effect of truncation is not significant (the figures are omitted).

A comparison between the Padé approximations and DNS can be also made in the wave-vector space. The coefficients $c_S^{(n)}(k)$, $n = 1, 2, 3, \ldots$, in the Taylor series (19) and (20), can be readily obtained from (16) through (21). The Eulerian DNS values of $R_E(k)$ are computed by Fourier transforming the Eulerian field $u(x,t)$. In isotropic turbulence, $c_S^{(n)}(k)$ and $R_S(k)$ depend on $k$ only through $k$, and in the following evaluations of such quantities by numerical
R̂ L

averaged Eulerian and Lagrangian spectra

Pade´ approximations in the wave number space for the band 

3 and 4 show comparisons between the DNS values and

in which

that the Fourier transform of

values of the Lagrangian correlations

~v̂

so that

from the DNS data by noting that

and that the effect of the truncation in wave-vector space may be

max

5 are labeled as DNS.

DNS values computed by doubling

K

max =21 are labeled as DNS.

In the passive vector equation, there is no viscous damp-

ing effect, in contrast to the Navier-Stokes equation (4), so that the effect of the truncation in wave-vector space may be more significant in (27) than in (4). In order to check this effect, we have solved (27) with doubling

K

max . Since there was no significant difference between the results obtained by the computations with

K

max =21 and 42, we omit plotting the DNS values computed by doubling

K

max in Fig. 4.

Approximations satisfying (a) and (b) are not limited to

the Padé approximation (22). For example, approximations

of the form

̂V̂(q,t) = ≤( ̂ψ(q,t;p,t0) ̂u(−p,t0))p−k.

Because of (9) and (10), its inverse Fourier transform

V̂k(x)

obeys the so-called passive vector equation

[∂ ̂u(x,t) · ∇x]V̂k(x,t) = 0,

and the following instantaneous condition

̂V̂k(q,t0) = ≤( 1/(2π)²δq+p ̂u(−p,t0))p−k,

where

δq+p = 1 if q + p = 0 and 0 otherwise. Thus we can estimate the correlation

R̂ L from (26) by simulating the Eulerian field

u

obeying (1) as well as the passive vector field

V

obeying (27) under the initial condition (28).

FIG. 3. The same as in Fig. 1, but for the normalized Eulerian two-time spectrum

R̂ E(t,k)=R̂ E(t0,k)/R̂ E(0,t0,k) for (a) k = 4 and (b) k = 8. The DNS values with

K

max =21 are labeled as DNS.

FIG. 4. The same as in Fig. 3, but for the normalized Lagrangian two-time spectrum

R̂ L(t,k)=R̂ L(t0,k)/R̂ L(0,t0,k).

simulations, we approximate them by their band average, i.e.,

⟨f(k)⟩ ~ ⟨f(p)⟩p−k,

in which

⟨f(p)⟩p−k denotes the band average of f over all wave vectors p satisfying

k − Δk/2 < |p| < k + Δk/2. Figures

3 and 4 show comparisons between the DNS values and Padé approximations in the wave number space for the band averaged Eulerian and Lagrangian spectra

R̂ E(t,k) and

R̂ L(t,k), respectively, where

R̂ E(t,k) = R̂ E(t0,k).

The DNS values of the Lagrangian correlations

R̂ L(t,k) are computed from the DNS data by noting that

̂ψ(−k,t0;t0) = ̂u(−k,t0),

and that the Fourier transform of (7) gives

⟨ ̂ψ(k,t0;t) · ̂ψ(−k,t0;t)⟩t0

= (2π)³ ∑

q

⟨ ̂u(−q,t) · ̂ψ(q,t;k,t0) · ̂u(−k,t0)⟩q.

(25)

so that (20), (25) and

R̂ L(t,k) = ⟨R̂ L(t0,k)⟩p−k yield

R̂ L(t,k) = ∑

q

⟨ ̂u(−q,t) · ̂V̂ E(q,t)⟩.

(26)

Here

̂ψ(q,t;k,t0)

is the Fourier transform of the Lagrangian position function

ψ(y,t;x,t0) and

̂V̂ E(q,t)

is defined by

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\[ R \approx \exp(\bar{P}), \quad \bar{P} = P_{K/M}(t) = P_K(t)/P_M(t), \]  
\hspace{1cm} (29)

can satisfy (a) and (b). The latter form is suggested by considering \( R_E \) in the case when the viscous effect is dominant, so that \( R_E(t,k) \sim \exp(-v_k \bar{t}) \). It is also suggested by the LRA, in which the closure equations can be simplified, for isotropic homogeneous and quasistationary turbulence, by introducing a function \( \phi \) defined by \( R_L(t,k) = \exp(-\phi) \). Note, however, that this form cannot represent negative values of \( R \), provided that \( \bar{P} \) is real.

One may also use the technique of expanding the Fourier transform of the correlation by use of a set of orthogonal functions as suggested by Kraichnan.\(^7\) Let \( F \) be related to \( R(t) \) as

\[ R(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega, \]

and be expanded as

\[ F(\omega) = \sum_{n=0}^{\infty} b_n P_n(\omega) w(\omega), \]  
\hspace{1cm} (30)

where \( P_n(\omega) \) are the set of polynomials orthogonal in \((-\infty, \infty)\) with respect to an everywhere positive weight \( w(\omega) \) such that

\[ \int_{-\infty}^{\infty} P_n(\omega) P_m(\omega) w(\omega) d\omega = \delta_{nm}, \]  
\hspace{1cm} (31)

and therefore,

\[ b_n = \int_{-\infty}^{\infty} \bar{F}(\omega) P_n(\omega) d\omega. \]

There are various possibilities in the choice of \( w(\omega) \). We consider here only one of the simplest choices, i.e., the case when \( w(\omega) \) is a Gaussian. If \( w(\omega) \approx \exp(-A\omega^2) (A > 0) \), \( P_n \) is a Hermite polynomial given by \( P_n(\omega) = C_n \exp(A\omega^2) \times ([dl/d\omega])^n \exp(-A\omega^2) \), where \( C_n \) is a normalization constant for \( P_n \) to satisfy (31), and the inverse Fourier transform of \( P_n(\omega) w(\omega) \) defined by \( (1/2\pi) \int P_n(\omega) \exp(-A\omega^2) \times \exp(-i\omega t) d\omega \) is shown to be proportional to \( r^n \xi(t) \) by partial integrations, where \( \xi(t) \) is the inverse Fourier transform of \( w(\omega) \approx \exp(-A\omega^2) \) and is proportional to \( \exp[-t^2/(4A)] \). The approximation for \( R(t) \) obtained by truncation of (30) to \( n \leq N \) is then given by

\[ R_N(t) = \exp\left( \frac{-t^2}{4A} \right) \left[ R(t) \exp\left( \frac{t^2}{4A} \right) \right]_{(N)}, \]
\hspace{1cm} (32)

where \([ \cdot ]_N\) denotes the truncation at term \( n^N \) of the Taylor series of the function within the brackets \([ \cdot ]_N\).\(^7\) Figures 1 through 4 show the approximations of \( R_0 \) and \( R_4 \) obtained from the Taylor series and (32) by using the Gaussian weight \( w(\omega) \) which makes \( b_2 = 0 \), i.e., \( A = -c_S^{(0)}/(2c_S^{(2)}) \) [or \(-c_L^{(0)}/(2c_L^{(2)}) \)]. The values of \( R_0 \) are the same as those of \( \exp(\bar{P}_{[02]}) \) with \( c_S^{(1)} \) ignored.

It is observed in the figures that even approximations as low as \( P_{[02]} \), \( \exp(\bar{P}_{[20]}) \), or \( R_0 \), give reasonably good agreement with DNS.

### IV. TIME MICROSCALES

The computation of coefficients such as \( \hat{c}_E^{(n)} \) and \( \hat{c}_L^{(n)} \) is in general much easier for lower \( n \). Moreover, the comparisons in Figs. 3 and 4 suggest that a rough estimate of the characteristic time scales of the Eulerian and Lagrangian two-point correlations may be obtained by knowing only a few lowest-order coefficients, e.g., those with \( n \leq 2 \). This is consistent with the inertial range behavior of homogeneous isotropic turbulence at infinitely large Reynolds number in the LRA, which gives for small \( t \),

\[ G(k,t) = \frac{\hat{R}_L(t,k)}{\hat{R}_L(0,k)} \sim 1 - \alpha \xi^2 t^2/2, \]
\hspace{1cm} (33)

whereas the numerical integration of the LRA equation gives the following integral time scale:

\[ T_L(k) = \int_0^\infty G(k,t) dt \sim \frac{2.16}{\sqrt{K}} \xi^{-1}, \]  
\hspace{1cm} (34)

in the inertial range.\(^15\),\(^16\) Here \( \epsilon \) is the energy dissipation rate per unit mass, \( K \) is the Kolmogorov constant and \( K = 1.72 \) according to the LRA. The lowest-order Padé approximation compatible with (33) is

\[ G(k,t) \sim (1 + \alpha \xi^2 t^2/2)^{-1}, \]
\hspace{1cm} (35)

which gives

\[ T_L(k) \approx \frac{2.16}{\sqrt{K}} \xi^{-1}, \]  
\hspace{1cm} (36)

while the Gaussian approximation compatible with (33) is

\[ G(k,t) \sim \exp(-\alpha \xi^2 t^2/2), \]
\hspace{1cm} (37)

which gives

\[ T_L(k) \approx \frac{1.22}{\sqrt{K}} \xi^{-1}. \]  
\hspace{1cm} (38)

Thus both approximations (35) and (37) give estimates of the integral time scale \( T_L(k) \) which are of the same order as the numerical solution (34). The comparison of (34), (36), and (38) shows that the Padé approximation overestimates, while the Gaussian approximations underestimate the integral time scale. A similar trend is seen in Fig. 4, where the decay of the Lagrangian correlation is slower in the approximation \( P_{[02]} \), while it is faster in \( \exp(\bar{P}_{[20]}) = R_0 \), as compared to the DNS values.

The coefficients \( \hat{c}_E^{(1)} \) and \( \hat{c}_L^{(1)} \) are negligibly small, when the homogeneous and isotropic turbulence is quasistationary in the sense that

\[ \frac{\partial}{\partial t} \left( \hat{u}(k,t+t_0) \cdot \hat{u}(\mathbf{r} = -k, t+t_0) \right) \bigg|_{t=0} = 2\hat{c}_E^{(1)}(k) \sim 0, \]
\hspace{1cm}
\[ \frac{\partial}{\partial t} \left( \hat{v}(k,t_0+t_0) \cdot \hat{v}(\mathbf{r} = -k, t_0+t_0) \right) \bigg|_{t=0} = 2\hat{c}_L^{(1)}(k) \sim 0. \]
We therefore consider the coefficients $\hat{c}^{(2)}_E$ and $\hat{c}^{(2)}_L$, which give the time microscales $\tau_E$ and $\tau_L$ defined as

$$\frac{\partial^2}{\partial t^2} \langle \hat{u}(k,t+t_0) \cdot \hat{u}(-k,t_0) \rangle \bigg|_{t=0} = -\frac{1}{[\tau_E(k)]^2} \langle \hat{u}(k,t_0) \cdot \hat{u}(-k,t_0) \rangle,$$

$$\frac{\partial^2}{\partial t^2} \langle \hat{v}(k,t_0;t+t_0) \cdot \hat{v}(-k,t_0;t_0) \rangle \bigg|_{t=0} = -\frac{1}{[\tau_L(k)]^2} \langle \hat{v}(k,t_0) \cdot \hat{v}(-k,t_0) \rangle,$$

i.e.,

$$[\tau_S(k)]^2 = -\frac{\hat{c}^{(0)}_L(k)}{\hat{c}^{(2)}_S(k)},$$

for isotropic turbulence, where we have used $\hat{c}^{(n)}_S(k) = \hat{c}^{(n)}_E(k)$ and $\tau_S(k) = \tau_E(k)$.

In contrast to higher-order derivatives, it is not difficult to compute the derivatives $\mathbf{u}^{(2)}$ and $\mathbf{v}^{(2)}$ according to (5) and (6) by using a database of DNS velocity fields as large as $512^3$ grid points for $\mathbf{v}^{(0)}$, and then to compute the coefficients $\hat{c}^{(2)}_E(k)$ and $\hat{c}^{(2)}_L(k)$ by (21). Figure 5 shows the time microscales of the Eulerian and Lagrangian two-time spectra obtained by using such a database, which will be referred to as Run512. The database gives a forced turbulence velocity field at time $t=12\tau$, where $\tau$ is one eddy turnover time, simulated under periodic boundary conditions from a random initial condition with forcing at wave numbers $1<k<2$ by an alias-free spectral method with $K_{max}=241$. The enstrophy of the simulated velocity initially increases and then starts to decay at about $12\tau$, when the microscale Reynolds number $R_k$ is about 126 and Kolmogorov dissipation wave number $k_d=(\epsilon/\nu^3)^{1/4}$ is about 101. (The DNS database was obtained under collaboration with Y. Yamazaki and its details will be presented elsewhere.) In Fig. 6 the values of energy spectrum at $t=12\tau$ are shown along with the values of $\hat{c}^{(2)}_E$ and $\hat{c}^{(2)}_L$. Time microscales are not defined for negative values of $\hat{c}^{(2)}_S(k)$. They are therefore omitted from the plots in Figs. 5 and 6.

It is observed in Fig. 5 that $\tau_S(k)$ scales as $k^{-2/5}$ in agreement with (33), while $\tau_E(k)$ scales with $k^{-1}$ at $k\sim 20$ or so, where the energy spectrum is seen in Fig. 6 to scale nearly as $k^{-5/3}$. The scaling $\tau_S(k) \propto k^{-1}$ is in agreement with the DIA for high Reynolds numbers, and may be explained by the so-called random sweeping effect or by assuming the dominance of nonlocal (in wave number space) interactions and statistical independence of small eddies from large eddies (cf. the next section as well as the following paragraphs). It also agrees with the numerical simulations by Sanada and V. Shanmugasundaram, and by Gotoh et al. Gotoh et al. also showed that the Lagrangian correlations decay on a time scale $1/\tau_S(E(p))p^3 dp$. Note, however, that the time dependence which they computed was the $t$-dependence in $\langle \hat{v}(k,t_0;t) \cdot \hat{v}(-k,t_0;t) \rangle$ which is used in the ALHDIA, but different from the $r$-dependence of (17) which is used in the LRA. The resolution or the Reynolds number of their simulation seems too low to exhibit the $k^{-2/3}$ dependence of the characteristic time scale of the Lagrangian correlation.

It is seen in Fig. 5 that $\tau_L(k)$ blows up rapidly with $k$ at high $k$, in contrast to $\tau_E(k)$. This is because the decrease with $k$ of $\hat{c}^{(2)}_L(k)$ in that wave number range is much faster than $E(k)$, in contrast to $\hat{c}^{(2)}_E(k)$, as seen in Fig. 6. In order to get some idea about the reason for such $k$-dependences of $\tau_S(k)$ and $\hat{c}^{(2)}_S(k)$, let us consider the case when the PDF of the velocity field is Gaussian with zero mean. By substituting $\mathbf{u}^{(2)}$ and $\mathbf{v}^{(2)}$ generated by (5) and (6) into $\hat{c}^{(2)}_S(k)$, it is shown that $\hat{c}^{(2)}_S(k)$ may be in general written as

$$\hat{c}^{(2)}_S(k) = C^{non}_S(k) + C^{min}_S(k) + C^{vis}_S(k),$$

where $C^{non}_S(k)$ ($C^{vis}_S(k)$) consists of terms, each of which contains only nonlinear (linear) operators and is expressed by a fourth (second)-order moment of $\mathbf{u}$, while $C^{min}_S(k)$ con-

FIG. 5. The Eulerian and Lagrangian micro time scales $\tau_S(k)$ and $\tau_L(k)$ for Run512. The values with $K_{max}=170$ are labeled 170. The straight dotted lines labeled 2/3 and 1, respectively, show the values $K^{-1}k^{-2/5}$ with $K = 2.28$ and 1/($U/k$).
sists of terms each of which contains one nonlinear and one linear operator, and is expressed by a third-order moment of \( \mathbf{u} \). Here

\[
C_{S}^{\text{vis}}(k) = \langle \left[ \hat{L}(\hat{\mathbf{u}}) \right](\mathbf{k}) \cdot \mathbf{u}(-\mathbf{k}) \rangle = -(v k^2)\mathcal{E}_{S}^{(0)}(k).
\]

Under the assumption of the Gaussianity of the PDF, the third-order moments are zero so that \( C_{S}^{\text{mix}}(k) = C_{E}^{\text{mix}}(k) = 0 \), and it is shown\(^{12,16} \) for isotropic (reflection invariant) homogeneous turbulence that

\[
\frac{1}{\tau_{E}(k)} = -\frac{\mathcal{E}_{E}^{(2)}(k)}{\mathcal{E}_{S}^{(0)}(k)} = B_{E}(k) - (v k^2)^2,
\]

where

\[
B_{E}(k) = k^2 \int_{0}^{\infty} h(q/k) E(q) dq = U^2 k^2 + k^2 \times \int_{0}^{\infty} h(q/k) - \frac{2}{3} ] E(q) dq
\]

and \( U^2 = (\mathbf{u} \cdot \mathbf{u})/3 \). Here \( E(k) = 2\pi k^2 \varepsilon_{S}^{(0)}(k) \), and \( h \) and \( j \) are positive functions of only \( q/k \). We omit here writing the expressions for \( j \) and \( h \), for which the reader may refer to Ref. 16. If the inertial range form \( E(k) \approx e^{\frac{2}{3}} k^{-\frac{5}{3}} \) is substituted, then both the integral (41) and the last one in (40) converge at both high and low wave numbers, and both of them scale like \( k^2 \mathcal{V}_{L}^2 \) in the inertial subrange, where \( \mathcal{V}_{L}^2 = E(k) k^2 \varepsilon_{S}^{(0)} k^{-\frac{2}{3}} \) \[cf.\ Ref. 16, where the factor \( h(1/k) \) in (A1) is a typographical error to be corrected to \( h(q/k) \).\]

It is shown that as \( k \to \infty \), the integral in (40) is much less than the first term \( U^2 k^2 \) on the r.h.s. of (40), and \( B_{E}(k) \) given by (41) tends to a constant of order \( \varepsilon/\nu \), for a wide class of functional forms of \( E(k) \) depending only on \( \varepsilon, \nu \) and \( k \) at the range of large \( k \) including the inertial range, as in Kolmogorov’s hypothesis. This suggests that \( 1/\tau_{E}(k) \to \frac{1}{U k} \) at large \( k \) provided \( (v k^2)^2 (U k)^{-1} \leq 1 \) (in the DNS this ratio is in fact small and about \( 0.03 \) at \( k \approx 100 \)), whereas the r.h.s. of (39) can be very small or even negative if \( (v k^2)^2 \) is comparable to the constant of \( \varepsilon/\nu \), which may occur at wave numbers of the order of the Kolmogorov dissipation wave number \( k_{\eta} = (\varepsilon/\nu^3)^{1/4} \). Thus \( \tau_{E}(k) \) may grow up sharply with \( k \) at large \( k \), in contrast to \( \tau_{E}(k) \). Note that the term \( (v k^2)^2 \) in (39) comes from the expansion of the viscous damping factor \( \exp(-v k^2 t) \) and it may be removed by considering the expansion of \( [R_{E}(t,k) \exp(-v k^2 t)] \) instead of \( R_{E}(t,k) \).

The above estimates for \( \mathcal{E}_{E}^{(2)}(k) \) and \( \mathcal{E}_{L}^{(2)}(k) \) are consistent with the DIA and the LRA, respectively, and give

\[
\tau_{E}(k) \sim \frac{1}{U k}
\]

and

\[
\tau_{E}(k) \sim e^{-1/3} k^{-2/3}/\sqrt{1.06k}
\]

in the inertial range [cf. (33)]. The values of (42) and (43) are shown by straight lines in Fig. 5, where the Kolmogorov constant is put at \( K = 2.28 \), which is obtained from the energy spectrum in Fig. 6 by a least square fit of \( E(k) k^{5/3} \) to \( K e^{2/3} \) in the wave number range \( 8 < k < 18 \). As noted above, the second term on the r.h.s. of (40) and the viscous effect may be so small as compared to the first term at large \( k \) that \( \tau_{E}(k) \) is well approximated by (42) not only in the inertial range but also in the dissipation range. On the other hand, the integral (41) does not scale as \( k^2 \mathcal{V}_{L}^2 \) in the dissipation range, so that the scaling implied in (43) does not hold in the range. In the inertial range, the ratio of the second to the first term of the r.h.s. of (40) is \( e^{-2/3} k^{4/3} / k^2 U^2 = O((k_{l}/k)^{2/3}) \), where \( k_{l} \) is the characteristic wave number of energy containing eddies defined by \( e = (\mathbf{u} \cdot \mathbf{u})^{3/2} k_{l}^{2} / 16 \). Hence the second term is negligible unless \( k/k_{l} \) is sufficiently small. Small deviation in Fig. 5 of the DNS values of \( \tau_{E}(k) \) from (42) at small \( k \) is consistent with this estimate.

Although it is implicitly assumed in the above theoretical estimates that the inertial range is infinitely wide, its finiteness, in addition to the non-Gaussianity, in the DNS may also affect the (dis)agreement between the theory and the DNS. The effect of the non-Gaussianity will be discussed in the next section. The discussion suggests that \( \tau_{E}(k) \) is less sensitive to the non-Gaussianity than \( \tau_{L}(k) \), and although the theory based on the Gaussian approximation may yield the proper scaling of \( \tau_{E}(k) \) in the inertial range, the quantitative agreement between DNS and the theory is not so good as that between DNS and the estimate (43) observed in Fig. 5, when the inertial range is finite (cf. Fig. 7).

Analyses of \( \mathcal{E}_{E}^{(2)}(k) \) similar to the above can be also made for rotationally invariant turbulence that is not reflection invariant as remarked in Ref. 12. A term was missed in the remark for the effect of the helicity on \( B_{E}(k) \) (private communication from Dr. Rubinstein). The reader may refer for an estimation of the effect of the helicity to calculations by Rubinstein and Ye Zhou.\(^{19} \)

The PDF of the velocity field in real turbulence is not Gaussian, and the third-order moments may be nonzero. Although it is therefore difficult to get rigorous estimates for \( \mathcal{E}_{E} \) and \( \mathcal{E}_{L} \), the above example suggests that \( B_{E}(k) \) can be very small, so that the r.h.s. of (39) can be even negative, and
can therefore be very large in contrast to $\tau_E(k)$. The results in Figs. 5 and 6 are consistent with this expectation. In particular, the values of $\hat{c}_E^{(2)}(k)$ obtained with $\nu = 0$, which are equal to those of $C_L^{\text{non}}(k)$, are seen to be very small at large $k$ in Fig. 6.

In order to check the possible effect of the truncation wave number $K_{\text{max}}$ in the operations $\mathcal{M}$ and $\mathcal{C}$ in (5) and (6), we have computed $\hat{c}_E^{(2)}(k)$ not only with the original values of $K_{\text{max}} = 241$ (Run512) but also with $K_{\text{max}} = 170$. The values of time microscales and $\hat{c}_E^{(2)}(k)$ thus computed are also shown in Figs. 5 and 6, respectively. They suggest that the effect is not significant.

If one would assume that $\langle \hat{v}(k,t_0; t) \cdot \hat{v}(-k,t_0; t) \rangle$ is independent of $t$ in stationary turbulence, then its differentiation twice with respect to $t$ gives

$$a(k) = \langle \hat{v}^{(1)}(k) \cdot \hat{v}^{(1)}(-k) \rangle$$

$$= - \langle \hat{v}^{(2)}(k) \cdot \hat{v}^{(0)}(-k) \rangle = - \hat{c}_L^{(2)}(k),$$

so that

$$A(k) = 2E(k)/[\tau_L(k)]^2$$

(44)

for homogeneous isotropic turbulence, where $A(k) = 4\pi k^2 a(k)$. However, the Lagrangian spectrum $\langle \hat{v}(k,t_0; t) \cdot \hat{v}(-k,t_0; t) \rangle$ need not be stationary even in the stationary turbulence, so that (44) need not hold even in stationary turbulence, and the difference of two curves $A(k)$ and $2E(k)/[\tau_L(k)]^2$ [the r.h.s. of (44)] obtained by using the DNS velocity field of Run512 as $u$ in (6) in Fig. 7 need not imply the nonstationarity of the DNS field. (Note: the integral of $\langle \hat{v}(k,t_0; t) \cdot \hat{v}(-k,t_0; t) \rangle$ over $k$ can be stationary. The Eulerian spectrum is stationary in stationary turbulence.)

If $\tau_L(k)$ scales like $\tau_L(k) \approx k^{-2/3}$ as seen in Fig. 5, and $E(k) = 2\pi k^2 \hat{c}_S^{(0)}(k) \approx k^{-5/3}$ then $E(k)/[\tau_L(k)]^2$ should scale as $k^{-1/3}$. Figure 7 suggests that the scaling of $E(k)/[\tau_L(k)]^2$ is not far from $k^{-1/3}$, but not close to that of $A(k)$. Note that if the viscous effect is indeed negligibly small in the inertial range, then $A(k)$ gives the spectrum of the pressure (gradient) correlation, whose scaling has attracted recent studies (cf., e.g., Nelkin and Chen\textsuperscript{20}).

Figure 8 shows the time microscales obtained from the DNS field (which is called here Run64) at moderate Reynolds number used in the previous section. It is seen that the Eulerian time microscale scales approximately like $\tau_E(k) \approx k^{-1}$ in a wide range of $k$ at the moderate Reynolds number, but the $k$-dependence of the Lagrangian microscale is not so simple. Values obtained by doubling $K_{\text{max}}$ (from 21 to 42) are also shown in Fig. 8, and again the difference seems to be insignificant, except at high wave numbers.

V. THE ROLE OF NON-GAUSSIANITY IN TURBULENCE

The multipoint joint probability distribution of the turbulence velocity field is known to be non-Gaussian due to the structure generated by the dynamics of fluid motion. It may be then natural to ask how important is the non-Gaussianity or the structure in turbulence.\textsuperscript{21,22}

In order to get some idea on the role of the structure in determining the Eulerian and Lagrangian two-time correla-

![Image](image-url)

**FIG. 8.** The same as in Fig. 5, but for Run64. The values with $K_{\text{max}} = 42$ are labeled as 42. [Micro time scales by artificial fields discussed in Sec. V; (a) randomly phase-shifted field $u^1$ (labeled as “shift”) and (b) random Gaussian fields $u^1$ and $u^2$ (labeled as “G1” and “G2”) are also plotted.]

![Image](image-url)

**FIG. 9.** Time microscales $\tau_E$ and $\tau_L$ based on the (a) original field $u$ (solid line), (b) randomly phase-shifted field $u^1$ (dashed line) and (c) random Gaussian field $u^2$ (dotted line) for Run512.
same range. This implies that the correlation \( c^{(2)}_E \) computed from the artificial fields is smaller than that by the DNS field. Note that if the viscous term is negligible, then \( c^{(2)}_E \) is given by a fourth-order moment of the velocity field. The figure therefore shows that the fourth-order moment is larger in the DNS field than in the randomly shifted or Gaussian field in the \( k \)-range under consideration. This is similar to the well known experimental observations that the flatness factor of the small scale velocity component is larger in turbulence than the Gaussian value of 3, and also that correlations such as the pressure correlation \( \langle \nabla p \cdot \nabla p \rangle \) may be underestimated by the use of artificial Gaussian fields.10

It is also observed in Fig. 9 that the difference between the values for the Eulerian time microscales by the three fields \( \mathbf{u} \), \( \mathbf{u}^E \), and \( \mathbf{u}^G \), are smaller than the corresponding difference for the Lagrangian time microscale. This implies that the Eulerian microscale \( \tau_E(k) \) and the derivative \( c^{(2)}_E \) are less sensitive than the Lagrangian microscale \( \tau_L(k) \) and derivative \( c^{(2)}_L \) to the possible structure (or non-Gaussianity) of turbulence. This may be understood as follows. If the Eulerian dynamics of small eddies is dominated by the interaction with energy containing large eddies, then the Navier-Stokes equation (1) in wave-vector space may be approximately written as

\[
\frac{\partial \mathbf{u}}{\partial t} \sim \mathcal{M} : \mathbf{u} \mathbf{u},
\]

under a symbolic notation where \( \mathbf{U} \) represents the energy-containing large eddies. Here any constant of order unity, as well as the viscous term, is ignored. The derivative \( c^{(2)}_E \) may then be approximated by

\[
\hat{c}^{(2)}_E \sim \langle (\mathcal{M} : \mathbf{U}) (\mathcal{M} : \mathbf{U} \mathbf{u}) \rangle \mathbf{u}.
\]

If the small eddies are statistically independent from the large eddies,16,23,24 then the fourth-order moment \( (\mathbf{U} \mathbf{U} \mathbf{u} \mathbf{u}) \) in (45) may be approximated by \( \langle \mathbf{U} \mathbf{U} \mathbf{u} \mathbf{u} \rangle \), which may be compatible with the statistics of the ensemble of the artificial randomly shifted or Gaussian random fields. Thus the microscale \( \tau_E(k) \) and the derivative \( \hat{c}^{(2)}_E \) may be insensitive to the difference between \( \mathbf{u} \) and \( \mathbf{u}^E \) or \( \mathbf{u}^G \).

The microscales computed similarly for Run64 are shown in Fig. 8. It also shows that the Lagrangian time microscales computed from the artificial fields are somewhat larger than the real ones, and the Eulerian time microscale is less sensitive than the Lagrangian microscale to the difference between \( \mathbf{u} \) and \( \mathbf{u}^s \) or \( \mathbf{u}^G \).

VI. DISCUSSION

It is shown in Secs. II and III that if one is given an Eulerian velocity field at an instant, and the dynamics, then one can generate the Taylor expansion series in powers of the time difference of the Eulerian and Lagrangian two-point two-time correlations, and Padé approximations that can be constructed from the Taylor series agree reasonably well with the full statistics computed from DNS of homogeneous and isotropic turbulence obeying the Navier-Stokes dynamics at moderate Reynolds number.

The DNS values of correlations studied in this paper have gentle time dependence and they become small for large \( t \). Such correlations are presumably favorable to the Padé approximations subject to the conditions (a) and (b) discussed in Sec. III, although rigorous mathematical proof for the validity of the Padé approximations is not known.10 Under certain dynamics, for example, in strongly stratified turbulence, the time dependence is expected to be oscillatory and not so gentle. The study of the applicability of Padé approximations to such cases is underway. It is in general difficult or very time consuming, if possible, to compute \( R_L(k) \) for all \( k \). The method of the Padé approximations presented in this paper enables us to efficiently compute the correlations that govern two particle diffusion, as well as Lagrangian autocorrelations that govern single particle diffusion,9 and it is hoped it will provide great practical utility in the study of turbulent diffusion.

The method of expansion is also applicable when only limited information rather than the full detail of the velocity field is available. For example, suppose that one is given only limited statistical information such as the energy spectrum, but one still needs to get estimates for the two-time correlations. Then one may first generate a set of random Gaussian fields \( \mathbf{u}^G \)'s compatible with the given spectrum, then substitute the random field \( \mathbf{u}^G \) into \( \mathbf{u}^{(0)} \) for computing.
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14The idea of using the passive vector equation to estimate the band averaged Lagrangian two-time correlation has been attempted previously by T. Gotoh (private communication).