Energy spectrum in the enstrophy transfer range of two-dimensional forced turbulence

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Numerical simulations of two-dimensional forced turbulence suggest that the enstrophy transfer range energy spectrum $E(k)$ a little steeper than $k^{-3}$ is robust in the sense that it may be realized in a wave number range under different run conditions. It is shown that such energy spectra fit well $E(k) = C_k \eta^{2/3} k^{-3} \left[ \ln(k/k_1) \right]^{-1/3}$, where $C_k$ is a dimensionless constant, $\eta$ the enstrophy transfer rate per unit mass and $k_1$ a wave number at the bottom of the range. The simulations give $C_k \approx 1.9$ in fairly good agreement with the existing theoretical estimates.

The study of two-dimensional (2D) turbulence has been stimulated by its possible relevance to global scale flows on the earth whose horizontal characteristic length scale is much larger than the vertical one. In 2D Navier–Stokes turbulence without external forcing, the total enstrophy as well as the total energy is conserved in the inviscid limit, and the enstrophy transfer rate or the enstrophy dissipation rate per unit mass $\eta$ is expected to play a key role in the dynamics at high Reynolds number. The dimensional analysis based on the wave number $k$ and $\eta$ suggests that there may exist an enstrophy-transfer range which exhibits the energy spectrum $E(k)$ of the form

$$E(k) = C \eta^{2/3} k^{-3},$$

where $C$ is a dimensionless constant. Kraichnan suggested that because of nonlocal interaction in $k$-space, Eq. (1) should be corrected to

$$E(k) = C_k \eta^{2/3} k^{-3} \left[ \ln(k/k_1) \right]^{-1/3} \quad (k \gg k_1),$$

where $k_1$ is a wave number at the bottom of the range. The constant $C_k$ has so far been estimated theoretically by the Test Field Model (TFM) and the Lagrangian Renormalized Approximation (LRA). The TFM gives $C_k = 1.74 g^{2/3}$, where $g$ is an adjusting parameter and the choice of $g = 1.064$ as in Ref. 4 gives $C_K = 1.82$, whereas the LRA, which contains no adjusting parameter, gives $C_K = 1.81$. There have been extensive studies to confirm or check the spectrum of Eq. (1) or (2) by numerical simulations. It has been reported that simulations with reasonably high-resolution (up to 512$^2$ grid points) of 2D turbulence forced at low-wave number range under periodic boundary conditions show the appearance of long-lived coherent vortical structures, and the observed energy spectrum is steeper than $k^{-3}$. Such coherent structures are known to also be manifested in decaying 2D turbulence. Numerical simulations with up to 4096$^2$ grid points resolution of forced 2D turbulence to study Reynolds number dependence of the spectrum in the enstrophy transfer range show that the spectrum consistent with Eq. (2) may be obtained at large enough microscale Reynolds number or by the use of hyperviscosity. Maltrud and Vallis showed numerically that when all coherent vortices are destroyed by strong infrared dissipation, a $k^{-3}$ spectrum in the enstrophy transfer range can be observed.

Recently we performed numerical simulations of 2D forced turbulence with resolutions up to 2048$^2$ grid points under periodic boundary conditions using a hyperviscosity, and also reanalyzed the data of energy spectra in previous studies; the spectral data were taken directly from the figures in Refs. 6 and 9 using an optical scanner and data-capturing software. The simulated spectra in general look steeper than $k^{-3}$ in the enstrophy transfer range. For such a spectrum, it might be tempting to try a fitting such as $E(k) \propto k^{-\alpha}$ presumably with $\alpha \neq 3$. It would then be difficult to derive theoretically the exponent $\alpha$ on the basis of the governing dynamics. However, an inspection of the simulation data suggests to try fitting with taking into account the logarithmic correction implied by the theoretical prediction Eq. (2), rather than fitting to $E(k) \propto k^{-\alpha}$ with noninteger exponent $\alpha$.

The aim of this Brief Communication is to show (i) taking into account the logarithmic correction may provide a
good agreement between the theory and simulated data, and (ii) the spectrum given by Eq. (2) is robust in the sense that a wave number range in which the energy spectrum $E(k)$ fits well Eq. (2) may be realized under different run conditions including those of our runs, Legras et al. and Borue. 

The governing equation in our simulations is given by

$$\frac{\partial \zeta}{\partial t} + J(\zeta, \psi) = F + D, \quad (3)$$

where $J$ is the Jacobian defined by $J(\zeta, \psi) = \frac{\partial \zeta}{\partial (x,y)} \frac{\partial \psi}{\partial (x,y)}$, and $\psi$ is the streamfunction related to the fluid velocity as $(u,v) = (\partial \psi / \partial y, -\partial \psi / \partial x)$.

The fundamental periodic box size is $2\pi$ in both $x$ and $y$ directions, and the number of the grid points in real space is set to 1024 or 2048 so that the retained wave number range is $k << K_{max}$ with $K_{max}$ = 481 or 963. The time advancing is done by a fourth-order Runge–Kutta method. To numerically simulate the inertial subrange of 2D turbulence the damping and forcing terms are modeled in the same way as Ref. 11, i.e., the dissipation function $D(k)$ in the wave vector space is given by

$$D(k) = -\alpha_k - \gamma \xi_{rms} \left( \frac{k}{K_{max}} \right)^{2n-2}, \quad (4)$$

in which the linear drag coefficient $\alpha$ is held constant in the wave number range $k < K_{\alpha}$ and set to 0 for $k > K_{\alpha}$. $\xi_{rms}$ is the r.m.s. vorticity which is calculated every time step, $\gamma$ is a tuning factor of order unity, and we set $n = 4$ for all simulations. For the forcing function $F(k)$, a random Markovian formulation is used, i.e.,

$$F_n(k) = A (1 - R^2)^{1/2} e^{i\theta} + RF_{n-1}(k), \quad (5)$$

where $F_n$ denotes the value of $F$ at the $n$th time step, $\theta$ is a random number in $[0, 2\pi]$ and differs for different $n$ and $k$, and the forcing amplitude $A$ is held constant for all the wave numbers satisfying $k \in (K_{min}, K_{max})$ and is set to zero for the other. $R$ is a function of the time increment $\Delta t$ and the forcing correlation time. The values of the parameters used in the run are listed in Table 1.

The initial condition for Run 1 was set to $\psi(x,0) = 0$, and the simulation was run until $t = 80$. The total energy and enstrophy of Run 1 initially increase with time and become almost stationary at $t = 40$, after which the energy spectrum and enstrophy flux $\eta(k)$ are also seen to be almost stationary, where $\eta(k)$ is computed as $\eta(k) = -\text{Re} \sum_{p \leq J(p)} \frac{1}{p} \times \xi(-p)$. The field $\psi$ of Run 1 at $t = 40$ was used as the initial condition for Run 2, and it was run until $t = 35$. The field of Run 2 becomes almost statistically stationary after $t = 20$.

Figures 1(a) and 1(b), respectively, show the simulated energy spectrum $E(k)$ and the enstrophy flux $\eta(k)$ averaged over the time interval $t = 70 - 80$ in Run 1 and $t = 25 - 35$ in Run 2, where the time intervals $10(t = 80 - 35$ and $35 - 25$) correspond to 42.7 and 35.6 in Run 1 and Run 2, respectively. Figure 1(b) shows that the enstrophy flux $\eta(k)$ is almost independent of $k$ in the wave number range (50,200) for Run 1 and (50,500) for Run 2 (the averaged values over these ranges are $\eta = 0.50$ and 0.26, respectively) and decreases with $k$ to zero for large $k$. The latter fact implies that the enstrophy is well dissipated for $k > 200$ in Run 1 and for $k > 500$ in Run 2, although the enstrophy dissipation range is not so clear in Fig. 1(a). [Our preliminary experiments suggest that clear enstrophy dissipation ranges can be observed in the energy spectra obtained by the runs with large values of $\gamma$, say 5 or so, but in those cases, the wave number ranges where $\eta(k) \approx 0$ are rather narrow in contrast to Run 1 and Run 2.] In the following, we analyze the energy spectra in Fig. 1(a), regarding the wave number range

<table>
<thead>
<tr>
<th>Run</th>
<th>$K_{max}$</th>
<th>$\Delta t$</th>
<th>$K_{\alpha}$</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$n$</th>
<th>$K_{min}$</th>
<th>$K_{max}$</th>
<th>$A$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run 1</td>
<td>481</td>
<td>0.001</td>
<td>25</td>
<td>6</td>
<td>1.0</td>
<td>4</td>
<td>10</td>
<td>14</td>
<td>1.0</td>
<td>0.5</td>
</tr>
<tr>
<td>Run 2</td>
<td>963</td>
<td>0.000</td>
<td>625</td>
<td>6</td>
<td>1.0</td>
<td>4</td>
<td>10</td>
<td>14</td>
<td>1.0</td>
<td>0.5</td>
</tr>
</tbody>
</table>
where $\eta(k) = \bar{\eta}$ as the inertial subrange at which the effect of the cutoff at $k = K_{\max}$ and that of the enstrophy dissipation mechanism should be small.

The energy spectra in Fig. 1(a) look slightly steeper than the KLB prediction (1). If one would assume $E(k) \propto k^{-\alpha}$ and make the least square fit of $[\ln k, \ln E(k)]$ of Run 1 to a straight line at the wave number range $k \in (50,200)$ for which $\eta(k) = \bar{\eta}$, then it gives $\alpha = 3.19$. The similar fitting for Run 2 at $k \in (50,500)$ gives $\alpha = 3.14$.

However, an inspection of the spectrum data shows that one need not introduce such a noninteger exponent, whose theoretical derivation is difficult. Figure 2 shows the plot of $[k^3 E(k)/\eta^{2/3}]^{-3}$ against $\ln k$. It is seen that $[k^3 E(k)/\eta^{2/3}]^{-3}$ is roughly proportional to $\ln k$ in the wave number range $(50,200)$ for Run 1 and $(100,400)$ for Run 2; the latter wave number range, which is somewhat arbitrary, is rather narrow in contrast to the inertial subrange of Run 2. Note that the spectrum Eq. (2) is the asymptotic form for $k/k_1 \gg 1$ and is derived by assuming the scaling range to be sufficiently wide. This implies that simulated spectrum does not necessarily fit Eq. (2) over the whole inertial range for such finite-size simulations as Run 1 and Run 2. The application of the chi-square fitting in the wave number range $k \in (50,200)$ for Run 1 gives $[k^3 E(k)/\eta^{2/3}]^{-3} = a + b \ln k$ (which implies $E(k) = \eta^{2/3} k^{-3} b^{-1/3} \left[ \ln(k/a) \right]^{-1/3}$) with $a = -0.452 \pm 0.047$ and $b = 0.157 \pm 0.005$, so that we have

$$C_K = 1.85(1.83 - 1.87), \quad k_1 = 17.8(14 - 23).$$

Similarly, we obtain

$$C_K = 1.97(1.95 - 1.98), \quad k_1 = 18.1(15 - 22)$$

for $k \in (100,400)$ of Run 2. Note that in the above fitting, $k_1$ and $C_K$ in Eq. (2) are not prefixed, but they are regarded as the parameters to be determined by the fitting. Of course, the values depend on the fitting range, but the dependence is not very strong, for example, $\{C_K, k_1\} = \{2.09,11\}$ for $k \in (80,320)$ and $\{1.83,29\}$ for $k \in (120,480)$. This dependence may be regarded as the reflection of the finiteness of the simulated inertial subrange. The values of the fitting functions $a + b \ln k$ and $C_K \eta^{2/3} k^{-3} [\ln(k/k_1)]^{-1/3}$ for Eqs. (6) and (7) over their fitting ranges are shown in Figs. 2 and 1(a), respectively. It is seen that the simulated energy spectra can fit well not only $E(k) \propto k^{-\alpha}$ but also Eq. (2), i.e., $E(k) = C_K \eta^{2/3} k^{-3} [\ln(k/k_1)]^{-1/3}$. Since the difference between the two functions over the same fitting range is smaller than the estimation error of $[k^3 E(k)/\eta^{2/3}]^{-3}$, it is not easy to determine which of the functions fits better the simulated spectra. However, it is worthwhile to note that the simulated values of $C_K \approx 1.8 - 2.0$ for Eq. (2) are in fairly good agreement with the theoretical values $C_K \approx 1.8,4,5$ and those of $k_1$ are consistent with the theories. It would be interesting to clarify the finite-size effect of the simulated inertial range and to verify the universality of the constant $C_K$ by the simulations with much higher resolutions.

The wave number dependence of the energy spectrum of stationary 2D turbulence steeper than $k^{-3}$ has been also observed in Legras et al.6 where it is reported that simulations of forced turbulence with $512^2$ grid points give stationary energy spectra that scale like $k^{-\alpha}$ in the enstrophy transfer range, in which the value of $\alpha$ depends mainly on forcing, and ranges from 3.5 to 4.2. However, the above results suggest us to reanalyse the data with taking into account the logarithmic correction. The application of the fitting procedure used in Fig. 2 to the data of Legras et al. (spectrum II in Fig. 1 of Ref. 6) then gives $E(k) \sim 35.0 k^{-3} (\ln(k/15.0))^{-1/3}$, the fitting is for $k \in (17,78)$, where $[k^3 E(k)]^{-3}$ is roughly proportional to $\ln k$. (Since there is no information about $\eta$ in Ref. 6, it is difficult to prefix the fitting range and determine the value of $C_K$.) It is seen in Fig. 3 that the simulated spectrum by Legras et al. fits well $E(k) = 35.0 k^{-3} (\ln(k/15.0))^{-1/3}$. We have not tried such a fitting to the other spectra reported in Ref. 6, i.e., their spectra I and III, because the inertial ranges of these spectra seem too narrow for the application of such a fitting.

Borue9 has also performed numerical simulations at high Reynolds number of forced turbulence with up to 40962 resolution. Some of the runs are based on the use of hyperviscosity and exhibit fairly wide wave number ranges where $\eta(k) = \text{const}$. He showed that the energy spectra then fit well the form Eq. (2) near the bottom of the enstrophy transfer range, and obtained the value $C_K \approx 1.5 - 1.7$. [In Ref. 9, the value 2.626 was cited from Ref. 4 as the theoretical value for $C_K$ by the TFM, but the value is to be corrected to 1.82 (for $g = 1.064$), see Ref. 14.] In the fitting, he plotted

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**Fig. 2.** Plots of the time-averaged values of $[k^3 E(k)/\eta^{2/3}]^{-3}$ for Run 1 and Run 2 as functions of $\ln k$. Thick lines denote the fitting functions $a + b \ln k$; $(a,b) = (-0.45,0.16)$ for Run 1 and $(-0.38,0.13)$ for Run 2.

**Fig. 3.** Comparison between the energy spectrum simulated by Legras et al. (dotted line, spectrum II in Fig. 1 of Ref. 6) and the log-corrected fitting function (thick solid line).
These spectra belong to a class of spectra different from that of Eq. (1) or (2), i.e., their appearance is due to a mechanism different from the one responsible for Eq. (1) or (2), and this class of spectra will be discussed elsewhere.

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5. Y. Kaneda, “Inertial range of two-dimensional turbulence in a Lagrangian renormalized approximation,” J. Fluid Mech. 30, 2672 (1987). [Expression (9) for a damping factor $\eta$ is too small by a factor of 2, so that $C_k$ is too small by a factor of $2^{2/3}$.