Energy Spectrum in the Near Dissipation Range of High Resolution Direct Numerical Simulation of Turbulence

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The energy spectrum in the near dissipation range of turbulence is studied by analyzing the data of a series of high-resolution direct numerical simulations of incompressible homogeneous turbulence in a periodic box with the Taylor micro-scale Reynolds number $R_l$ and resolution ranging up to about 675 and 4096, respectively. The spectra in the Reynolds number range fit well to the form $C(k\eta)^n \exp(-\beta k\eta)$ in the wavenumber range $0.5 \lesssim k\eta \lesssim 1.5$, where $\eta$ is the Kolmogorov dissipation length scale and $C$, $\alpha$ and $\beta$ are constants independent of $k$. The values of $\alpha$ and $\beta$ decrease monotonically with $R_l$, and they are consistent with the conjecture that they approach to constants as $R_l \rightarrow \infty$, but the approach, especially that of $\beta$, is slow.

KEYWORDS: homogeneous isotropic turbulence, high-Reynolds-number turbulence, direct numerical simulation, energy spectrum, near dissipation range

1. Introduction

The energy spectrum of the fluctuating velocity field is one of the most simple and fundamental measures characterizing the statistics of fully developed turbulent flows that consist of a wide range of scales, so-called eddies. The spectrum has long been the subject of extensive studies, and it is widely assumed in the studies that in fully developed turbulence at sufficiently large Reynolds number $Re \gg 1$, the spectrum at sufficiently large wave number, or small scales, takes an universal form that is insensitive to the details of the boundary conditions and initial conditions, etc., at large scales.

In particular, according to the Kolmogorov hypotheses (K41),1 the energy spectrum takes the form

$$E(k) = \frac{(\langle v^2 \rangle)^{1/3}}{C} F(k\eta)$$

in the wavenumber range $k \gg k_l \equiv 1/L$, the so-called universal equilibrium range, of fully developed turbulence, where $L$ is the characteristic length scale of the energy containing eddies, $\langle v \rangle$ the mean rate of energy dissipation per unit mass, $\eta$ the Kolmogorov dissipation length scale defined by $\eta = (\langle v^2 / \nu \rangle)^{1/4}$, and $\nu$ the kinematic viscosity.

The assumption that $E(k)$ becomes independent of $\nu$ for $k\eta \ll 1$, yields $F(x) \sim x^{-5/3}$ for $x \ll 1$, so that

$$E(k) \sim K_0 (\langle v^2 \rangle) \eta^{2/3} k^{-5/3},$$

in the inertial subrange (ISR), $k_l \ll k \ll k_d \equiv 1/\eta$, where $K_0$ is a universal non-dimensional constant and $k_d = 1/\eta$ the Kolmogorov wavenumber.

K41 implies that the spectrum is universal not only in the ISR, but also in the dissipation range (DR), $k \gtrsim k_d$ for sufficiently large $Re$. The idea of universality of the energy spectrum plays important roles in many turbulence theories and modeling, and has attracted many studies. The reader may refer for the studies to the reviews by Nelkin,2) and Sreenivasan & Antonia3) and references cited therein.

In the studies, the $k^{-5/3}$ spectrum in the ISR at $Re \gg 1$ has been supported, at least approximately, by experiments and direct numerical simulation (DNS) of turbulence, (but also see Kaneda et al.,4) for recent DNS results). However, as compared to the ISR, relatively little seems known about the spectrum in the DR at large $Re$, although in the spirit of K41, the effect of non-universality at large scales is weaker in the DR than in the ISR.

Regarding the energy spectrum in the DR, it is also worthwhile to note that the skewness $S$ of longitudinal velocity derivative in stationary homogeneous isotropic turbulence can be expressed as an integral of $k^2 E(k)$ over $k$ under an appropriate normalization,5) and the integral is shown to be dominated by the contribution from the DR. Recent experiments have shown that $S$ increases with the Taylor scale Reynolds number $R_l$ for $R_l > 300$ or so, which suggests that the energy spectrum in the DR depends on $R_l$, i.e., on $Re$, in the Reynolds number range.

These considerations motivate us to study the spectrum in the DR at large $R_l$. Regarding the form of $E(k)$ or $F(k\eta)$ in the DR, a number of proposals have been so far made. Among them, a widely proposed one is

$$E(k) / (\langle v^2 \rangle)^{1/3} = C (k\eta)^n \exp[-\beta (k\eta)^n],$$

where $C$, $\alpha$, $\beta$ and $n$ are constants.

If one assumes that interactions distant in the wavenumber space are dominant, so that the small scale fields may be approximated as collections of small eddies under random large-scale straining fields, and the dynamics may be linearized in a certain sense at the scales, then one obtains $n = 2$, as shown by Townsend.6) One might think that since the velocity amplitude at small scale is small, such a linearization can be well justified.

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However, a family of two-point closure theories, including Quasi-Normal Approximation, Direct Interaction Approximation, Lagrangian History Direct Approximation, and Lagrangian Renormalized Approximation, suggest that it is not the case, and they give \( n = 1 \) for the far dissipation range, i.e., \( k\eta \gg 1 \). The value \( n = 1 \) has been also supported by the other theories and experiments. The form (3) with \( n = 1 \) has been supported also by direct numerical simulations (DNSs) of turbulence, although the DNSs have so far been limited only up to \( R_\lambda \sim 100 \) (see Martínez et al., and references cited therein).

To the authors’ knowledge, there has been no rigorous theoretical derivation of the form \( F \) in the DR, and it seems unlikely possible in near future. It is then reasonable to first observe the data of the spectrum by experiments and DNS, independently from any particular theory. It is however difficult to measure experimentally the spectrum at high wavenumber, because the signal level is so weak as compared to the background noise at these scales. On the other hand, DNS of turbulence provides detailed data free from experimental uncertainties.

It is to be recalled here that even if the assumption of the universality is asymptotically valid for \( Re \to \infty \), \( Re \) cannot be infinite in real turbulence. In this respect, little seems known quantitatively on the implication of the condition “\( Re \gg 1 \)”, or the \( Re \)-dependence of the spectrum. What is lacking here is the quantitative understanding of the condition “\( Re \gg 1 \)”. To get some idea on the \( Re \)-dependence at large \( Re \) by DNS, one need simulate turbulence with \( Re \) or \( R_\lambda \), as high as possible, whereas the resolution or the attainable Reynolds number is limited by the obvious limitation of available computer speed and memory, or resource. Regarding the examination of the form (3) in the DR or the near dissipation range of incompressible turbulence by DNS, they have been so far limited to resolution up to only 512 \(^3 \) grid points or \( R_\lambda \lesssim 100 \), as noted above.

This situation has been considerably changed by the appearance of the Earth Simulator (ES), which has the peak performance of 40 Tflops, and main memory of about 10 Tbyte. We could recently have a chance to carry out a series of DNS of incompressible turbulence obeying the Navier–Stokes equations with resolution up to 4096 \(^3 \) on the ES. In this paper, we study the the energy spectrum in the DR in the light of the data obtained by the DNS on the ES.

Before going to the analysis of the DNS data, it may be worthwhile to remark here a limitation of DNS, which arises from the simple fact that the number of total grid points in any DNS is limited by the available computer speed and memory, or resource. Under a given constraint of the number, (i) the requirement of simulating high \( Re \) turbulence conflicts with (ii) that of keeping large \( k_{\max} \eta \), where \( k_{\max} \) is the maximum wavenumber in the DNS. If one wishes to satisfy (i) or (ii), then one has to sacrifice the other, as will be explained in §2.

This is also true, even if one can use the ES. For example, \( k_{\max} \eta \) is limited only up to 2 or so, in order to perform DNS with \( R_\lambda \sim 700 \) on the ES. Since our main interest is in the possible universality that is expected to appear only at high \( Re \), we have to sacrifice (ii) to some extent. We therefore confine ourselves in this paper to the analysis of the near dissipation range, where \( 0.5 \lesssim k\eta \lesssim 1.5 \). Analysis using deeper dissipation range would be interesting, but it would require further computational capability and resource.

2. DNS Method and Run Conditions

A series of simulations of incompressible turbulence obeying the Navier-Stokes equations has been carried out in a periodic box with sides \( 2\pi \). The simulations use an alias free spectral method, and the time was advanced by a 4th-order Runge-Kutta method. The minimum wavenumber \( k_{\min} \) as well as the wavenumber increment is 1, and the total kinetic energy \( E \) of the fluctuating velocity was kept almost time independent (~0.5) by introducing negative viscosity in the wavenumber range \( k < 2.5 \). The numerical method of the DNS is basically similar to that in our previous studies Preliminary report of the DNS with an emphasis on the aspect of parallel computing on the ES was presented in ref. 18. In all the runs reported below except Run4096-2, we used double precision arithmetic, while in Run4096-2, we also used double precision arithmetic for the convolution sums in evaluating the nonlinear terms in the wavevector space, but used single precision arithmetic for time integration.

In order to choose a proper value for the kinematic viscosity \( \nu \) in the DNS, it is to be recalled that \( k_{\max} / k_{\min} = k_{\max} \ll N / 2 \), and

\[
k_{\max} / k_{\min} = k_{\max} \eta / k_{\min} \eta = (k_{\max} \eta) \times L_K / \eta,
\]

where \( L_K = 1 / k_{\min} \) and \( N \) is number of grid points in each direction of the Cartesian coordinate, so that \( N^3 \) is the total number of the grid points. According to K41, we have \( L / \eta = O(Re^{3/4}) \), and therefore the constraint for \( k_{\max} \eta \) such that

\[
N \approx (k_{\max} \eta) \times O(Re^{3/4}) \approx (k_{\max} \eta) \times O(Re^{3/4}),
\]

where we have used \( L \approx L_K \). This implies that if one wishes to simulate turbulence at high \( Re \) for given \( N \), then one cannot put \( k_{\max} \) too large. In the runs reported below, except Run512-4, the maximum wave number \( k_{\max} \) and \( \nu \) were so chosen that \( k_{\max} \eta \approx 2 \).

Some DNS parameter values and turbulence statistics are summarized in Table I, where \( R_\lambda \) is the Taylor micro-scale Reynolds number defined by \( R_\lambda = u' \lambda / \nu, \lambda = (15 u'^2 / \langle \epsilon \rangle)^{1/2} \) the Taylor micro-scale, \( 3 u'^2 / 2 = E \), and \( L \) the integral length scale defined as

\[
L = \frac{\pi}{2 u'^2} \int_{k_{\max}}^{k_{\max}} E(k) / k \ dk.
\]

Since \( u'^2 = 2 E / 3 \) and \( E = 0.5 \) in our runs, one eddy-turn-over time \( T = L / u' \) is about 2, in each run. Figures 1(a) and 1(b) show the DNS values of \( \langle \epsilon \rangle(t) \) and \( R_\lambda(t) \). It is seen that \( \langle \epsilon \rangle(t) \) and \( R_\lambda(t) \) are almost stationary at \( t = t_\text{t} \), except for Run4096-2, which is so huge simulation that it could be run only up to time \( t_\text{t} = 3.8 \). Here \( \langle \epsilon \rangle(t) \) and \( R_\lambda(t) \) are to be understood as the instantaneous values before taking the time average.

3. Energy Spectrum in the Near Dissipation Range

Figure 2 shows the energy dissipation spectrum \( D(k) = k^2 E(k) / \nu \) versus \( k \eta \). Here and hereafter, we use statistics averaged over the time span from \( t_\text{t} - 2 \) to \( t_\text{t} \), with an
Table I. DNS parameters and turbulence characteristics at the final time \( t = t_f \). \( N \): number of grid points in each direction of the Cartesian coordinate, \( \Delta t \): time increment, \( \lambda \): Taylor micro-length scale.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( R_L )</th>
<th>( k_{\text{max}} )</th>
<th>( 10^3 \Delta t )</th>
<th>( 10^4 \nu )</th>
<th>( \langle \epsilon \rangle )</th>
<th>( L )</th>
<th>( \lambda )</th>
<th>( 10^4 \eta )</th>
<th>( t_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run256-2</td>
<td>256</td>
<td>94</td>
<td>121</td>
<td>1.0</td>
<td>20</td>
<td>0.0936</td>
<td>1.10</td>
<td>0.326</td>
<td>17.1</td>
</tr>
<tr>
<td>Run512-4</td>
<td>512</td>
<td>94</td>
<td>241</td>
<td>20</td>
<td>0.0942</td>
<td>1.10</td>
<td>0.326</td>
<td>17.1</td>
<td>2</td>
</tr>
<tr>
<td>Run512-2</td>
<td>512</td>
<td>173</td>
<td>241</td>
<td>7.0</td>
<td>0.0795</td>
<td>1.21</td>
<td>0.210</td>
<td>8.10</td>
<td>10</td>
</tr>
<tr>
<td>Run1024-2</td>
<td>1024</td>
<td>268</td>
<td>483</td>
<td>2.8</td>
<td>0.0829</td>
<td>1.12</td>
<td>0.130</td>
<td>4.03</td>
<td>10</td>
</tr>
<tr>
<td>Run2048-2</td>
<td>2048</td>
<td>429</td>
<td>965</td>
<td>1.1</td>
<td>0.0824</td>
<td>1.01</td>
<td>0.0817</td>
<td>2.00</td>
<td>10</td>
</tr>
<tr>
<td>Run4096-2</td>
<td>4096</td>
<td>675</td>
<td>1930</td>
<td>0.44</td>
<td>0.0831</td>
<td>1.05</td>
<td>0.0515</td>
<td>1.01</td>
<td>3.8</td>
</tr>
</tbody>
</table>

Fig. 1. Time dependence of (a) \( \langle \epsilon \rangle(t) \) and (b) \( R_L(t) \). The symbols \( \square \), \( \bigcirc \), \( \Delta \), \( \triangledown \) and \( \phi \) denote the instantaneous values at \( t = 6, 7, 8, 9 \) and \( 10 \), respectively, and \( \square \) denotes the value at \( t = 3.8 \) for Run4096-2.

appropriate time interval (with time increment \( \Delta t = 0.1 \)) if not otherwise stated. This implies that we use time averaged dissipation rate \( \langle \epsilon \rangle \) in estimating the Kolmogorov dissipation length scale \( \eta = (\nu^3/\langle \epsilon \rangle)^{1/4} \) as well as the Taylor micro-scale Reynolds number \( R_L = u' \lambda / \nu \) with \( \lambda = (15 \nu u^2/\langle \epsilon \rangle)^{1/2} \).

The curves in Fig. 2 appear to overlap fairly well in the low wavenumber range, say \( 0.01 < k \eta < 0.1 \). It is also seen that the dissipation peak \( L_\eta \) decreases monotonically with \( R_L \), and the corresponding wavenumber is \( k_\eta \approx 0.17/\eta \), respectively, of \( R_L \) in the \( R_L \) range studied here. If we convert numerically our three-dimensional spectrum \( E(k) \) into \( E_{1d}(k) \) using the relation

\[
E_{1d}(k) = \frac{1}{2} \int_k^\infty \left( 1 - \frac{k^2}{k^2} \right) \frac{E(k)}{k} \, dk,
\]

and then measure the wavenumber \( k_\eta \) at which \( k^2 E_{1d}(k) \) attains its peak, we obtain \( k_\eta \approx 0.11 \). This value 0.11 is consistent with the experimental values of \( k_\eta \in (0.09, 0.15) \) plotted in Fig. 3 of ref. 19 for \( E_{1d}(k) \) and \( R_L \) ranging from 130 to 13,000.

Figure 3 shows DNS values of

\[
\ln[E(k + 1)/E(k - 1)]/\ln(k + 1)/(k - 1)
\]

as functions of \( k \eta \), for the energy spectrum \( E(k) \) used in Fig. 3. Equation (5) is a finite difference approximation of

\[
A(k) \equiv d \ln E(k)/d \ln k.
\]

From Fig. 3, one can see that \( A(k) \) may be well approximated by a linear function of \( k \eta \), such as

\[
A(k) = \alpha - \beta k \eta,
\]

at high wavenumber range, say \( k \eta \gtrsim 0.5 \), except near \( k \eta = 2 \), \( k \eta \gtrsim 1.7 \), where the data are contaminated by the effect of discarding high wave number modes in the DNS. This form (7) is consistent with eq. (3) with \( n = 1 \), and the constants \( \alpha \) and \( \beta \) can be estimated from the DNS data in the range.

For such an estimate, Martinez et al.\(^{17} \) assigned local

Fig. 2. Normalized energy dissipation spectra \( D(k) = k^2 E(k) \) vs \( k \eta \) averaged over the time \([t_f - 2, t_f]\). The inset shows a close-up view near the dissipation peak.

Fig. 3. The same as in Fig. 1, but for \( d \ln E(k)/d \ln k \) vs \( k \eta \).
values of $\alpha$ and $\beta$ at $k\eta$ by using the least-square fits of their DNS values of $A(k)$ to $\alpha - \beta k\eta$ in the range $[k\eta - d, k\eta + d]$, where $2d$ is the wavenumber interval for the fitting and they put $d = 0.9$.

Here we use a method basically similar to, but a little different from theirs. We use the data of $E(k)$ directly, rather than $A(k)$, and try the least-square fit of $\alpha \ln k\eta - \beta k\eta + c$ to the DNS values of $\ln E(k)/(\epsilon\nu^2)^{1/4}$ in the range $[k\eta - d, k\eta + d]$, where $d$ is set to be 0.5 in this paper. In the present method, one need not estimate the derivative $A(k)$, and can still obtain the estimates for $\alpha$, $\beta$ and $C(= \exp c)$ in eq. (3), as well as their standard errors (see, for example, §15.4 of ref. 20). We also tried the method by Martinez et al. (but with $d = 0.5$), and confirmed that there is no significant difference regarding the values of $\alpha$ and $\beta$ between the two methods.

Martinez et al. carried out DNS with deep dissipation range, $k_{\text{max}} \eta$ up to 12, but with low $R_1$, (for example, $R_1 = 38$ for the DNS with $k_{\text{max}} \eta = 12$). For the examination of the spectrum form such as (3) or the estimate of the constants in the expression, it is certainly desirable to use data of deep enough dissipation range. However, in order to simulate high $Re$ turbulence under given number of $N^3$, one need sacrifice keeping high $k_{\text{max}} \eta$ to some extent, as explained in previous sections. We put in the present study $k_{\text{max}} \eta \sim 2$ in view of the constraint (4), and confine ourselves in the following to the analysis of the near dissipation range of turbulence with $R_1$ up to about 675.

Before investigating the values of $\alpha$, $\beta$ and $C$ on the basis of the present DNS data, it is desirable to have some idea on the effect of our choosing $k_{\text{max}} \eta \sim 2$. For this purpose, we performed a preliminary run, Run512-4, in which we set $k_{\text{max}} \eta \sim 4$ instead of 2. Its initial field was set to be the same as the final field of Run256-2, i.e., the field at $t = 10$, We then carried out Run512-4 until $t = t_f = 2$, and confirmed its energy spectrum for $k\eta > 2$ to be almost stationary for $t \gtrsim 1$.

Figure 4 compares the values of $A(k)$ for Run256-2 and Run512-4 obtained by the values of $E(k)$ averaged over the time span from $t_f - \tau$ to $t_f$ with every time interval $\Delta \tau = 0.1$, where $\tau = 2$ and 1 for Run256-2 and Run512-4, respectively. It shows that there is no significant difference between the values for $A(k)$ by these runs in the wavenumber range, say $k\eta < 1.7$. Figure 5 compares the values of $\alpha$, $\beta$ and $C$ for these runs, assigned by the method noted above. It is seen that the difference between these runs is small in the range $k\eta \lesssim 1.3$. These results suggest that the effect of truncation such as $k_{\text{max}} \eta \sim 2$ is not significant for the local values of $\alpha$, $\beta$ and $C$ in the range $k\eta < 1.3$ or so. These facts provide us with a support of our analyzing the spectra in the range $k\eta < 1.3$ or so, by using the data of DNS with $k_{\text{max}} \eta \sim 2$.

It can be also observed in Fig. 5 that the values of $\alpha$, $\beta$ and $C$ around $k\eta \sim 2$ by the DNS with $k_{\text{max}} \eta \sim 4$ are approximately $k$-independent, and they are not very different from the values at, say, $k\eta = 1$, by the DNS with $k_{\text{max}} \eta \sim 2$. Such $k$-independence of $\alpha$ and $\beta$ at $1 \lesssim k\eta \lesssim 3$ was also observed by Martinez et al. (but with $d = 0.5$), although the values of $\alpha$ and $\beta$ as well as the Reynolds numbers are different from the present ones.

As a demonstration of the degree of the performance of the fitting to the DNS data by the function form (3), Fig. 6(a) shows the values of $A(k)$ by the DNS and the fitting for the representative case Run4096-2, where the values of $\alpha$, $\beta$ and $C$ assigned at $k\eta = 1$ are used. Figure 6(a) also includes the comparison for Run512-4. Similarly, Fig. 6(b) shows the comparison for $D(k)$. These figures show that the function form (3) fits well in the near dissipation range $0.5 < k\eta < 1.3$ and that the difference between these runs is small in the range $k\eta \lesssim 1.3$. These results suggest that the effect of truncation such as $k_{\text{max}} \eta \sim 2$ is not significant for the local values of $\alpha$, $\beta$ and $C$ in the range $k\eta < 1.3$ or so. These facts provide us with a support of our analyzing the spectra in the range $k\eta < 1.3$ or so, by using the data of DNS with $k_{\text{max}} \eta \sim 2$.
1.5 to the DNS data.

In contrast to the range $0.5 < k\eta < 1.5$, in the low wavenumber range the fit is seen to be not so good in Fig. 6. This is not surprising because the present fitting based on eq. (3) is only for the near dissipation range $0.5 < k\eta < 1.5$. If one is interested in the fitting for a wider range including the inertial subrange, one may consider, for example, composite forms such as those proposed by Sirovich et al., Lohse and Müller-Groeling, etc., instead of eq. (3). One might be then interested in the performance of the fitting based on such function forms in the near dissipation range.

As a typical example of such functions, let us consider the one discussed by Lohse and Müller-Groeling, which may be derived from the following approximation for the second order velocity structure function proposed by Batchelor:

$$D(r) = \frac{r^2}{(3\eta^2)} \left[ 1 + (1/3b) \right] (r/\eta)^2 [1 - 1/2]$$

where $D(r)$ is defined by

$$D(r) = \langle [u(x + r) - u(x)]^2 \rangle.$$

Equation (8) yields the energy spectrum of the form

$$E(k) = \frac{1}{2\pi} \int_0^\infty kr \sin(kr)D(r)dr$$

$$\propto \xi^{5/2} K_{3/2+\xi/2}(\tilde{k}) + \tilde{k}^{3/2-\xi/2} K_{1/2+\xi/2}(\tilde{k}).$$

Here $\tilde{k} = kr_d$, $r_d = (3b)^{3/4}$ and $K_{\nu}(\tilde{k})$ is the modified Bessel function of order $\nu$. In eq. (8), $b$ and $\xi$ may be determined, for example, by the least square fit of the logarithm of the right-hand side of eq. (8) to the DNS values.

Along with the DNS values and the fitting based on (3), Fig. 6 also shows the fitting thus obtained by using eq. (9) and the DNS data for $0.2 \leq \log_{10}(r/\eta) \leq 2.5$ of Run4096-2, where $b = 6.68$ and $\xi = 0.718$, and the amplitude of the fitting spectrum is adjusted so that the peak value of $k^2E(k)$ agrees with the DNS data. Figure 6 shows that the agreement between the fitting and the DNS data is not very much improved by the use of the form (9), as compared with the fitting based on the simpler non-composite form (3), at least in the near dissipation range. The similar is also the case for the fitting based on the form proposed by Sirovich et al.

She and Jackson suggested that the experimental values of $(k/k_p)^2 E_{id}(k)/E_{id}(k_p)$ collapse well to a universal curve, when plotted as functions of $\xi = k/k_p$. Such plots collapse fairly well as seen in Fig. 7, where we plot the DNS values of $(k/k_p)^2 E(k)/E(k_p)$ as a function of $\xi = k/k_p$. However, note that the collapse of the curves to a universal curve implies that $G$ is a function of only $\xi$, independent of $R_s$, so that $A(k)$ defined by eq. (6), i.e.,

$$A(k) = d \ln E(k)/d \ln k = -2 + d \ln G(\xi)/d \ln \xi,$$

depends only on $\xi$. This means that if we plot $A(k)$ as a function of $\ln k\eta$ as in Fig. 3, then the slopes of the curves must be the same to each other. However, this conflicts with Fig. 3, which shows different slopes for different $R_s$, i.e., the DNS data suggest that the slope, which corresponds to $\beta$ in eq. (7), depends on $R_s$.

She and Jackson also proposed an empirical fit for the whole range of (one-dimensional) energy spectrum in the following composite form

$$E_{id}(k) = E_{id}(k_p)[\xi^{\alpha/3} + \alpha \xi^{-\beta}] \exp(-\mu \xi),$$

where $\xi = k/k_p$, and $\alpha = 0.8$, $\beta = 1$ and $\mu = 0.63$ were determined from the experimental data. To test the performance of the fit, we convert $E_{id}(k)$ given by eq. (10) into $E(k)$ as before, and plot $(k/k_p)^2 E(k)/E(k_p)$ as a function of $k/k_p$ in Fig. 7, where $k_p$ is the wavenumber at which $k^2 E(k)$ peaks. The result shows that the agreement between the fitting and the DNS data is fairly good but not very much improved by the use of eq. (10), as compared with eq. (3), at least in the
near dissipation range. The similar was also the case for the composite form (9).

In this paper, we confine ourselves to the near dissipation range, and consider the fitting based on (3) in the following.

4. $R_d$-Dependence of the Constants

Figure 8 shows the values of $\alpha$, $\beta$ and $C$ assigned at $k\eta = 1$, from the spectra $E(k)$ used in Fig. 3. Figures 8(a) and 8(b) also include the data by Martinez et al.\textsuperscript{17} for $R_d \leq 102$. They argued that values of $\alpha$ and $\beta$ at $k \lesssim 4/\eta$ may be different from those at $k \gtrsim 4/\eta$, and estimated them for $k\eta < 3$, and $k\eta > 6$, separately (Fig. 3 in ref. 17). The values in the former group are used in Figs. 8(a) and 8(b).

It is seen in Figs. 8(a) and 8(b) that the values of $\alpha$ and $\beta$ in the present study ($94 \leq R_d \leq 675$) are consistently smaller than those in their DNS ($R_d \leq 102$). But the trend in the latter, i.e., the lower $R_d$ group appears to continue smoothly to that in the former, i.e., higher $R_d$ group. The present data together with those by Martinez et al. suggest that $\alpha$ and $\beta$ monotonically decrease with $R_d$. In the present DNS, $(\alpha, \beta) \approx (-2.0, 4.5)$ at $R_d \approx 94$, and $(\alpha, \beta) \approx (-2.5, 3.5)$ at $R_d \approx 675$. The speed of decrease with $R_d$ slows down with $R_d$.

According to K41, if $E(k)$ is given by the form (3), then $\alpha$, $\beta$ and $C$ must be universal constants independent of $R_d$ for sufficiently large $R_d$. This implies

$$\alpha \to \alpha_\infty, \beta \to \beta_\infty, C \to C_\infty,$$

as $R_d \to \infty$, i.e., there are non-dimensional universal constants, $\alpha_\infty$, $\beta_\infty$ and $C_\infty$, such that $a$, $b$, $c$ defined by

$$\alpha = \alpha_\infty + a(R_d), \quad \beta = \beta_\infty + b(R_d), \quad C = C_\infty + c(R_d), \quad (11)$$

tend to 0, as $R_d \to \infty$.

The data shown in Fig. 8 are consistent, or at least not inconsistent, with this conjecture, and it is tempting to try a simple power law fitting such as

$$a(x) = \xi_a x^{-n_a}, \quad b(x) = \xi_b x^{-n_b}, \quad c(x) = \xi_c x^{-n_c}, \quad (12)$$

where $n_a$, $n_b$, $n_c$ and $\xi_a$, $\xi_b$, $\xi_c$ are constants to be determined by the fitting. The least-square fitting of the forms (11) with (12) to the DNS data plotted in Fig. 8, which include those by Martinez et al., gives

$$\alpha_\infty = -2.9, \quad \beta_\infty = 0.62, \quad C_\infty = 0.044, \quad (13)$$

and

$$a(x) = 7.3x^{-0.47}, \quad b(x) = 9.3x^{-0.19}, \quad c(x) = 24.2x^{-0.43}. \quad (14)$$

The solid lines in Fig. 8 are the fitting curves, and are seen to be in good agreement with the DNS data. It is to be noted that the fitting parameters may be sensitive to the choice of the data. In fact, if one uses in the fitting only the present DNS data with excluding the data by Martinez et al.,\textsuperscript{17} then one would obtain values quite different from eqs. (13) and (14).

The agreement of the fitting curves and the DNS data by itself does not exclude the possibility that functions different from eq. (12) may also fit well to the DNS data. The implication of the fitting in Fig. 8 is to be understood, not as a verification of the universal validity of (11) with (13) and (14), but as a suggestion that the convergence of $\alpha$, $\beta$ and $C$ to the constants $\alpha_\infty$, $\beta_\infty$ and $C_\infty$, as $R_d \to \infty$, if they converge, are so slow that $\alpha$, $\beta$ and $C$ can be close to $\alpha_\infty$, $\beta_\infty$ and $C_\infty$ only at very large $R_d$. For example, according to eqs. (11), (13) and (14), even at $R_d$ as large as $R_d = 10,000$, the differences $\alpha - \alpha_\infty$, $\beta - \beta_\infty$ and $C - C_\infty$ are still as large as about 3, 261 and 105% of $|\alpha_\infty|$, $\beta_\infty$ and $C_\infty$, respectively.

5. Discussion and Conclusion

The analysis in §4 is based on time-averaged data. On the other hand, if we understand the average $\langle \ldots \rangle$ as the instantaneous average in an appropriate sense, such as ensemble or space average, before taking the time average, the statistics may depend on time. In fact, a close inspection of the basic statistics such as the mean rate of energy dissipation $\langle \epsilon \rangle$ and Taylor micro-scale Reynolds number $R_d$ reveals that they are not time-independent in a strict sense, although the time dependence is not so strong after $t > 5$ (see Fig. 1). Then, for example, $\langle \epsilon \rangle$ and the Kolmogorov...
dissipation length scale $\eta = (\langle v^3 / \epsilon \rangle)^{1/4}$ in eq. (1) are to be understood as $\eta = \eta(t)$ and $\langle \epsilon \rangle = \langle \epsilon \rangle(t)$.

In this regard, it may be worthwhile to quote here the following well-known comment to K41 by Landau:25) (The symbol $\ell$ and $\epsilon$ in the comment correspond to the present $L$ and $\langle \epsilon \rangle$, respectively.)

"... The instantaneous values of $\langle (v_2 - v_1) (v_2 - v_1) \rangle$ might in principle be expressed as a universal function of the energy dissipation rate $\epsilon$ at the instant considered. When we average these expressions, however, an important part will be played by the manner of variation of $\epsilon$ over times of the order of the periods of large eddies (with size $\sim \ell$), and this variation is different for different flows. The result of averaging therefore cannot be universal."

When applied to eq. (1), this comment implies that
\[
\frac{(\langle \epsilon \rangle) v^3)^{1/4} F(kn)}{\epsilon} \neq \langle \epsilon \rangle v^3)^{1/4} F(kn)
\]
and that the function form (1) cannot be universal if $\langle \epsilon \rangle$ in eq. (1) is to be understood as the time averaged dissipation rate. Here the bar $\overline{\cdots}$ denotes the time average, and $n\rangle = (\langle \epsilon \rangle)^{1/4}$.

Because the dissipation rate fluctuates not only in time, but also in space, it is tempting to reinterpret Landau’s comment in the context of spatial fluctuation or intermittency. As a matter of fact, it is common to do so, as in the revision of K41 by Kolmogorov himself.25) Since the dissipation rate, averaged over the spatial domain of a flow, depends on the large-scale statistics, it follows that the form (1) as well as the constant $K_n$ in (2) cannot be universal if they are asserted for flows with arbitrary large-scale statistics.26)

However, as noted by Kraichnan,26) (i) K41 is intended to describe a universal statistical state attained by small scales, and therefore it should be applied only to subregions of a flow sufficiently small compared with gross dimensions that cascading has been able to set up that state, and (ii) K41 is not logically disqualified merely because the dissipation rate fluctuates.

In order to get some idea on the possible effect of the time-dpendence, we plot in Fig. 8 in addition to the values obtained in the previous sections, the instantaneous values for the constants ($\alpha$, $\beta$, $C$) at $t = 6, 7, 8, 9$ and 10, where the constants are estimated by using the instantaneous values of $E(k)$, $\langle \epsilon \rangle$, $\eta$ and $R_1$. Figure 8 shows that although the constants at different time do not completely agree with each other, the difference among them is not so large, and the instantaneous values of the constants are not far from the averaged ones. The effect of the time averaging on the constants is small provided that variations of $E(k)$, $\langle \epsilon \rangle$, $\eta$ and $R_1$ in time are small as in our DNS.

The scatter of the values of the constants at different time shown in Fig. 8 suggests that the first sentence in Landau’s comment cited above is not strictly satisfied. One might think that Fig. 8 therefore disqualifies the universality of eq. (1). However, the dissipation rate $\langle \epsilon \rangle(t)$ in our DNS is slightly and slowly fluctuating as seen in Fig. 1, and the DNS results do not rule out the possibility that the cascading has been able to set up, only approximately, the universal equilibrium state at small scales assumed in K41. The scatter in Fig. 8 by itself therefore does not disqualify the universality of eq. (1) for the possible universal equilibrium state.

It is outside the scope of the present paper to identify the origin of the scatter, or to discuss for example whether it is because the statistical state at small scales is not close enough to the universal equilibrium state assumed in K41. In authors’ understanding, what is learned from Fig. 8 is that the difference of the constants at different time is small, i.e., the values at different time collapse well to the extent as shown in Fig. 8. In this regards, it is to be recalled that experiments and DNS so far made suggest that the intermittency correction, even if it exists, to the inertial range spectrum (2) is small and support the universality of the constant $K_n$, at least approximately, although they do not support K41 concerning higher order statistics.27)

It is difficult to conclude, from only the results of the previous section, the universality of the value of $\alpha$ nor $\beta$, nor the function form (3) with $n = 1$. However, in the cases considered in the present paper, the function form (3) fits well to the DNS data to the degree as demonstrated in Figs. (6(a) and 6(b), and the time-dependence, nor the scatter of the measured values for ($\alpha$, $\beta$, $C$) for given resolution is not so large (see Fig. 8).

In conclusion, our DNS data suggest, within the Reynolds number range studied here ($R_i \lesssim 675$),

(i) the form (3) with $n = 1$ continues to be a good approximation with the DNS at $R_i \gtrsim 100$,

(ii) the values $\alpha$ and $\beta$ decrease monotonically with $R_i$, and

(iii) the DNS data are consistent with the conjecture that $\alpha$ and $\beta$ tend to constants independent of $R_i$ as $R_i \rightarrow \infty$, but the approach is slow as discussed in §4.

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1) A. N. Kolmogorov: C. R. Acad. Sci. URSS 30 (1941) 301.


